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FOUNDATIONS

Quine, W. V. On relations as coextensive with classes. J. Symbolic Logic 11, 71–72 (1946).

In a previous paper [same J. 10, 95–96 (1945); these Rev. 7, 185] Quine presented a definition of the ordered pair $(x; y)$ of entities x and y in such a way that $(x; y)$ is of the same logical type as x and y. This provided a simplification of definitions due to Wiener and Kuratowski. In any system in which this definition may be carried out, every entity in the fundamental domain must be regarded as a class. The present note points out the further property that every entity in the fundamental domain must now also be regarded as an ordered pair. Hence the distinction between classes and relations is dropped altogether, and laws heretofore holding only for relations now hold generally.

R. M. Martin (Bryn Mawr, Pa.).

Hiż, Henri. Remarque sur le degré de complétude. C. R. Acad. Sci. Paris 223, 973–974 (1946).

For a deductive system of propositions X, the author indicates the following inductive definition: X is complete to degree 1 in case there exists no meaningful proposition independent of X; X is complete to degree α in case there exist exactly $\alpha-1$ meaningful nonequivalent propositions which are independent of X and such that on the adjunction of one of them to X the resulting system is complete to degree $\alpha-1$. Let I be the system of propositions represented by the set of those formulas containing only propositional variables and $\supset$ which are provable in the intuitionistic propositional calculus. The author gives rules for the construction of an infinite sequence of propositions such that on the adjunction of the nth of these to I, a system complete to degree $n+1$ results. Hence the system I is said to be complete to degree infinity. Similar results have been obtained by Gödel [Anz. Akad. Wiss. Wien. Math.-Nat. Kl. 69, 65–66 (1932)] and Jaskowski [Actes du Congrès International de Philosophie Scientifique, Paris, 1935, v. 6 [Actual. Sci. Indus., no. 393, Hermann, Paris, 1936], pp. 58–61].

D. Nelson (Washington, D. C.).

Dijkman, J. G. Einige Sätze über mehrfach negativkonvergente Reihen in der intuitionistischen Mathematik. Nederl. Akad. Wetensch., Proc. 49, 829–833 = Indagationes Math. 8, 532–536 (1946).

The paper is based on Belinfante's extension [same Proc. 33, 1170–1179 (1930)] of a definition of Brouwer [J. Reine Angew. Math. 154, 1–7 (1925)]. A sequence $\{s_n\}$ is said to be negatively convergent (mehrfach negativ-konvergent) to $\alpha_1, \cdots, \alpha_N$ in case, for every $\epsilon>0$, there does not exist any monotone increasing sequence of integers $n(i, k)$ such that $|s_{n(i,k)}-\alpha_i|>\epsilon$ for every k and every i, $1\leqq i\leqq N$. Simple negative convergence of $\{s_n\}$ to α is a special case of the definition for $N=1$. The definition may be extended for convergence to an ω-sequence $\{\alpha_i\}$. In case s_n is the nth partial sum of a series S, S is said to be negatively convergent to $\alpha_1, \cdots, \alpha_N$. Among the author's results are the following. If $\{\alpha_i\}$ is negatively convergent to α, and if any S is negatively convergent to α or to $\alpha_1, \cdots, \alpha_N, \alpha$, where $\alpha_1, \cdots, \alpha_N$ is a subsequence of $\{\alpha_i\}$, then S is negatively convergent to $\{\alpha_i\}$. If any S is negatively convergent to $\{\alpha_i\}$, and if $\{\beta_i\}$ is negatively convergent to α_1, then S is negatively convergent to the point set $\{\beta_1, \beta_2, \cdots, \alpha_2, \alpha_3, \cdots\}$. The author also considers conditions under which convergence of S to an ω-sequence may be replaced by convergence of S to a finite sequence.

D. Nelson.

Goodstein, R. L. Function theory in an axiom-free equation calculus. Proc. London Math. Soc. (2) 48, 401–434 (1945).

This paper is concerned with a mathematical system which can be described roughly as combining features of the Hilbert-Bernays elementary calculus with free variables [Grundlagen der Mathematik, v. 1, Springer, Berlin, 1934, pp. 286–383] with those of a system due to Curry [Amer. J. Math. 63, 263–282 (1941); these Rev. 2, 340]. Curry's theory is an equational formulation of (primitive) recursive arithmetic in which the propositional calculus is not presupposed as an underlying logical calculus but is rather set up within the system in terms of certain arithmetic functions. Goodstein develops in some detail what is substantially this system. The principle of induction is rendered superfluous without essential weakening of the system. His theory appears adequate for most of elementary number theory. The extension of the system to nonnegative rationals is considered, as well as certain constructive portions of the theory of real numbers.

R. M. Martin.

Blake, Archie. A Boolean derivation of the Moore-Osgood theorem. J. Symbolic Logic 11, 65–70 (1946).

This paper is concerned with a direct (as opposed to indirect) proof of the Moore-Osgood theorem (that if a function $f(x, y)$ converges uniformly in x with respect to y, and converges in y for every x, then the function converges with respect to x and y jointly). It is shown that the essential (in a not too clear meaning of "essential") steps in the proof may be carried out utilizing only an applied functional calculus of first order.

As the author remarks, a full analysis of this proof could not be given wholly upon the basis of first order logic, since such concepts as equality and limit are employed fundamentally. However, the author conjectures that the manipulations whose difficulty retarded the discovery of this theorem belong wholly to first order logic. If this is true, we are provided with a striking argument as to the practical utility of a thorough knowledge of even elementary logic for the working analyst.

R. M. Martin (Bryn Mawr, Pa.).

Carnap, Rudolf. Remarks on induction and truth. Philos. and Phenomenol. Res. 6, 590–602 (1946).

Recalling the general burden of the Symposium on Probability, parts I and II [cf. these Rev. 7, 186–192] the author,

in this part III, notes a far-reaching degree of agreement, both in the general empiricist spirit and in the disposition to admit, in addition to frequencies, a conception of probability of a logical order, necessary for induction. He notes, however, that Reichenbach and von Mises take exception, each in his own way, with the last point.

Residual differences between the author and Nagel, Kaufmann, and Williams, while profitable to discuss because of their general common ground, are not pursued in the present paper, with the exception of the following two points between the author and Kaufmann. (1) On the nature of inductive inference. Whereas Kaufmann agrees with Carnap that in deduction (L-implication) the relation is purely logical (analytic), depending on the analysis of meanings of terms, and that only in applications do questions of the general body of knowledge (including synthetic, that is, observational material) come into play, on the other hand, Kaufmann regards inductive inference (degree of confirmation, "probability") as essentially different in all these respects. Carnap disagrees with this view, and holds that the relation between a sentence and a second one induced from it with a given probability is of a logical nature, granted the definition of "degree of confirmation"; he regards Kaufmann's insistence that an induction is made only relatively to a body of accepted knowledge as applying not to the logical process of induction, but to its application, where, indeed, a similar state of affairs applied to L-implication. And Carnap questions Kaufmann's statement that, in contradistinction from deductive logic, the complete formulation of the inductive relation between two sentences must explicitly refer to some "presupposed rules of induction." Finally, he rejects Kaufmann's idea of a sharp distinction between "accepted" and "rejected" sentences in a scientific enquiry: to Carnap, there should be a shading, more of a "probability," with which such sentences are held. (2) The concept of truth. Carnap regards Kaufmann as confusing "truth" and "knowledge of truth" (or verification). This he illustrates by familiar examples.

We may note as particularly relevant to the subject of the Symposium that in his article I.6 [p. 594], Carnap's example of degree of confirmation contains the sentence "If e and nothing else is known by X at time t, then h is confirmed by X at t to the degree 2/3." And he observes that while X may in effect have more knowledge than e, such knowledge is supposed to be "irrelevant" for h. There is here the usual pitfall presented by the use of the idea "relevance" regarded as self-explanatory concept in an analysis of probability. *B. O. Koopman* (New York, N. Y.).

Kaufmann, Felix. On the nature of inductive inference. Philos. and Phenomenol. Res. 6, 602–609 (1946).

The paper is essentially an attempt to answer Carnap's criticisms of Kaufmann's views [cf. the preceding review]. The author's first point is that Carnap's idea that "imperfect knowledge" (synthetic propositions held only tentatively) can approximate more and more to "perfect knowledge" (those held definitively and without prospect of later rejection) implies the, to Kaufmann, inacceptable notion of perfect knowledge as an empirically meaningful concept. It would violate the "principle of permanent control," according to which every synthetic proposition should be capable in principle of being rejected at some time or other. The author thus regards Carnap's conception of truth of synthetic propositions as having no procedural significance, and therefore as having no place in the analysis of inductive inference. Kaufmann believes that the general tendency to pattern the idea of truth of synthetic propositions upon that of analytic ones is an unfortunate one. The bearing of all this upon Carnap's earlier objections to the author's position is discussed.

The author reasserts his dichotomy between "accepted" and "rejected" propositions as basic to his interpretation of inductive inference. He explains that he does not conceive of the logic of science as a description of what the scientist actually does, irrespective of the level of clarity at which he proceeds, but rather as affording a system of standards of scientific criticism, standards which we have to make explicit.

Finally, the author objects to Carnap's parallelism between deductive and inductive reasoning, since he regards Carnap to have incorrectly used the concept of "truth" as the same in each case. Passing Carnap's linguistic system of induction in review, he objects that it is not the one used by scientists, that its conclusions are relative to a choice of the "range," a choice which may always have to be modified, thus depriving the conclusions of the absolute character of an L-implication; and he states: "The decisive difference between the two cases is that given the propositions i and j, it is determined, irrespective of any further rules, whether i entails [L-implies] j, whereas it is not determined, irrespective of further rules, whether i confirms j to a certain degree."

The fact that nobody seems to have worked out a generally satisfactory scheme of rules for induction seems to be the main general import of the latter comments for the subject of the Symposium. *B. O. Koopman.*

Carnap, Rudolf. Rejoinder to Mr. Kaufmann's reply. Philos. and Phenomenol. Res. 6, 609–611 (1946).

The author passes briefly in review, with added clarity, the main points at which he and Kaufmann are in agreement or disagreement. *B. O. Koopman* (New York, N. Y.).

von Mises, Richard. Comments on Donald Williams' reply. Philos. and Phenomenol. Res. 6, 611–613 (1946).

A brief statement of the author's strong disagreement with Williams' views of probability and interpretation of the opinions of the other members of the Symposium. The author also expresses doubts concerning Carnap's "probability$_1$." He makes no points not already familiar in his writings. *B. O. Koopman* (New York, N. Y.).

Nagel, Ernest. Is the Laplacean theory of probability tenable? Philos. and Phenomenol. Res. 6, 614–618 (1946).

The well-known objections to the Laplacean theory are mentioned in connection with Williams' attempt to assert this theory as the one giving the only true meaning to "probability." Vaguenesses and other difficulties with Williams' thesis are set forth. The essential ideas have occurred in earlier papers in the symposium.
B. O. Koopman (New York, N. Y.).

Williams, Donald. The problem of probability. Philos. and Phenomenol. Res. 6, 619–622 (1946).

After repeating the case for a probability in a sense of graded credibility, for regarding it as a logical notion, and for its further study, the author makes a series of brief remarks intended to answer Nagel's and some of von Mises' objections to his more or less Laplacean views. There seems

to be nothing essentially new in this last paper of the Symposium, not occurring earlier in it. *B. O. Koopman.*

Andreoli, Giulio. Interpretazione probabilistica di teorie logiche e matematiche relative a fenomeni concreti. Rend. Accad. Sci. Fis. Mat. Napoli (4) 12, 245–250 (1942).

Bouligand, G. La mathématique, science des problèmes. Rev. Gén. Sci. Pures Appl. **53**, 118–124 (1946).

Langhaar, Henry L. A summary of dimensional analysis. J. Franklin Inst. **242**, 459–463 (1946).

The idea that is emphasized in the following treatment is that dimensional analysis may be regarded as the mathematical theory of a class of functions that is characterized by a generalized type of homogeneity. It is felt that this point of view most clearly exhibits the scope of dimensional analysis, and that it avoids philosophical questions that have led to much controversy. *Author's introduction.*

ALGEBRA

de Bruijn, N. G. A combinatorial problem. Nederl. Akad. Wetensch., Proc. **49**, 758–764 = Indagationes Math. **8**, 461–467 (1946).

The number 00010111, considered as a repeating segment of an infinite series, contains all 8 triples of digits 0 and 1, as does its reverse 11101000. These are the only arrangements of 8 digits each 0 or 1 with this property. Using properties of finite graphs the author proves a conjecture of K. Posthumus (relative to a problem in telephone communication which is not described) that the number of such arrangements of 2^n digits each either 0 or 1 is $2^{f(n)}$, $f(n)=2^{n-1}-n$. The graphs concerned are oriented and have n vertices, $2n$ edges, each vertex having 2 edges pointing away and 2 towards it (a closed loop at a vertex is taken as having two edges oppositely directed). A doubling of a graph is defined as a transformation changing edges into vertices and vice versa: two vertices are connected in the double when their corresponding edges in the original terminate in opposite sense on a vertex. A "complete walk" being defined as a line passing through each edge of a graph once and only once, it is shown that doubling a graph of n vertices multiplies its number of complete walks by 2^{n-1}, from which the result in question follows by proper labelling of the graphs. *J. Riordan* (New York, N. Y.).

Martino, Caio Manlio. Un triangolo aritmetico relativo a una questione di formule ricorrenti. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) **4(73)**, 716–722 (1940).

The two-variable case of Waring's formula in the theory of symmetric functions is proved by induction. *J. Riordan* (New York, N. Y.).

Martino, Caio Manlio. Estensione del campo dei coefficienti binomiali dal triangolo di Tartaglia al piano cartesiano. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) **5(74)**, 305–309 (1941).

The binomial coefficient ${}_{-n}C_m$ is defined in the usual way as $(-1)^m{}_{n+m-1}C_m$. *J. Riordan* (New York, N. Y.).

Finney, D. J. Orthogonal partitions of the 6×6 Latin squares. Ann. Eugenics **13**, 184–196 (1946).

Continuing his research on the orthogonal partitions of a 6×6 Latin square [same Ann. 12, 213–219 (1945); these Rev. 7, 107], the author considers each of the seventeen types of square as listed by Fisher and Yates [Statistical Tables, 2d ed., Oliver and Boyd, London, 1943; these Rev. 5, 207]. He finds, e.g., that types IV, V, VI, X, XV, XVI admit all the nine partitions $(1,5)$, $(1^2,4)$, $(1^3,3)$, $(1^4,2)$, $(1^2,2^2)$, $(1,2,3)$, (3^2), (2^3), $(2,4)$. On the other hand, types XI, XII, XIII, XIV, XVII admit the last two only: in each case at least 216 (2^3) partitions and at least 81 $(2,4)$ partitions. Certain pairs of square-types are considered together as being "conjugate," so the final table contains 108 entries, of which 40 (like those mentioned above) are "probably not the full total." *H. S. M. Coxeter* (Notre Dame, Ind.).

Silva, Giovanni. Una generalizzazione del problema delle concordanze. Ist. Veneto Sci. Lett. Arti. Parte II. Cl. Sci. Mat. Nat. **100**, 689–709 (1941).

The problem considered is that of coincidences in a game of patience called counting by m's, played with a deck of cards of r colors with cards of each color numbered 1 to n. Counting by m's means counting out numbers 1 to m and repeating as often as necessary until the deck is laid out; a coincidence is scored when the number of the card laid out and the number called agree. The probabilities of exactly c, or at least c, coincidences are required. The author first notes that both are determined (by the method of inclusion and exclusion) when certain sums S_i (closely related to factorial moments) are known. For $ms=nr$ and $m\geqq n$, a complicated formula for S_i is used to determine the following approximations for the probability of exactly c coincidences:

$$(*)\qquad P_c\sim\frac{s^ce^{-s}}{c!}\left\{1-\frac{r-1}{2sm}\,\frac{(s-c)^2-c}{s}\right\},$$

$$P_c\sim\frac{s^c}{c!}\left(1-\frac{1}{n}\right)^{sn}\left\{1+\frac{1}{2m}+\frac{(r-1)(2s+1-c)c}{2s^2m}\right\}.$$

No mention of the extensive American and English literature on card matching appears. Kullback [Ann. Math. Statistics 10, 77–80 (1939)] has shown the limiting distribution to be Poisson, the first term of (*). *J. Riordan.*

Cattaneo, Paolo. Sul problema delle concordanze. Ist. Veneto Sci. Lett. Arti. Parte II. Cl. Sci. Mat. Nat. **101**, 89–104 (1942).

Silva's results in the problem of coincidences [cf. the preceding review] are extended to terms of order n^{-2} as follows:

$$P_c\sim\frac{s^ce^{-s}}{c!}\left\{1-\frac{r-1}{2}\left(\frac{1}{rn}+\frac{1}{r^2n^2}\right)\frac{(c-s)^2-c}{s}+\frac{r-1}{24r^2s^2n^2}f_c\right\},$$

$$f_c=8(r-2)\{(c-s)^3-3c(c-s)+2c\}$$
$$\qquad+3(r-1)\{(c-s)^4-6c(c-s)^2+c(11c-8s)-6c\}.$$

[Reviewer's note: the accuracy of this result may be improved by replacing $N^{-1}+N^{-2}$, $N=rn$, in the second bracketed term by $(N-1)^{-1}$ and N^{-2} by $\{N(N-1)\}^{-1}$ in the last, as emerges naturally in a process suggested by Kaplansky and Riordan, Ann. Math. Statistics 16, 272–277 (1945); these Rev. 7, 309.]

In a historical note, the author finds the problem for $m=n$ posed by E. Lemoine in 1899 in L'Intermédiare des Mathématiciens and answered for $c=0$ by L. Lindelöf [Öfversigt af Finska Vetenskaps-Societetens Förhandlingar 42, 79–87 (1900)]. In his note Lindelöf uses a method of multiplication of equal symbolic (rectangular) polynomials akin to that used more generally by Kaplansky [Amer. Math. Monthly 46, 159–161 (1939)]. *J. Riordan.*

Cattaneo, Paolo. Sul problema delle concordaaze generalizzato. Ist. Veneto Sci. Lett. Arti. Parte II. Cl. Sci. Mat. Nat. 103, 439–456 (1944).

The card-matching problem treated by Silva and Cattaneo [cf. the two preceding reviews] is generalized to two-deck matching for decks of arbitrary specification, that is, for any number of colors or kinds, and any number of cards of each color. The resulting formulas are identical with those obtained by the symbolic method [I. Kaplansky, Bull. Amer. Math. Soc. 50, 906–914 (1944); these Rev. 6, 159]. *J. Riordan* (New York, N. Y.).

Neuhaus, F. W. Affektlosigkeit der Gleichungen für fast alle Werte des linearen Koeffizienten. Deutsche Math. 7, 87–116 (1942).

Consider an algebraic equation

$$f_\nu(x, t)=x^n+a_1x^{n-1}+\cdots+a_{\nu-1}x^{n-\nu+1}+tx^{n-\nu}$$
$$+a_{\nu+1}x^{n-\nu-1}+\cdots+a_n=0,$$

where the a_i are fixed integers and t is a parameter, which is "without affect," that is, has a Galois group of order $n!$ in the field $P(t)$ of all rational functions of t with rational coefficients. From Hilbert's irreducibility theorem it follows that $f_\nu(x, t)=0$ will still be without affect for an infinite number of special integral values of t. From a theorem of Dörge [Math. Ann. 95, 84–97 (1925)] it follows that the set of special values of t for which the equation is not without affect is of small "density" (in a certain sense) in the set of all integers. This exceptional set is, however, in general infinite. The author studies the question of the conditions under which it becomes finite. A very general result is obtained for $\nu=n-1$, namely: if $a_n\neq 0$ then $f_{n-1}(x, t)=0$ remains not without affect for at most a finite number of special values of t. For $\nu=n$ the result is that $f_n(x, t)=0$, when $n>9$ is composite, remains not without affect for at most a finite number of special values of t, apart from the infinite set of special values of t for which x is rational. An earlier result of the author [Deutsche Math. 1, 519–524 (1936)] was that $f_\nu(x, t)=0$ is without affect in $P(t)$ for all but a finite set of values of another coefficient a_μ with $(\mu, n)=1$. Combining this result with those of the present paper it appears that an equation without affect can be constructed by the choice of only two of the coefficients and that even in their selection only a finite set of integers need to be avoided. The proofs use group theory, in addition to theorems on Diophantine equations due to Mordell and Siegel. *O. Todd-Taussky* (London).

Lipka, Stephan. Die Descartessche Zeichenregel und interszendenten Polynome. Deutsche Math. 7, 83–85 (1942).

An "interszendentes" polynomial is a function

$$f(x)=a_0x^{\alpha_0}+a_1x^{\alpha_1}+\cdots+a_nx^{\alpha_n}.$$

It is assumed that $\alpha_0<\alpha_1<\cdots<\alpha_n<1$, that the coefficients $\{a_i\}$ are real numbers and that $x^\alpha=\exp(\alpha\log x)$, where $\log x$ denotes the principal value. Let p, q, r denote the number of positive, negative and complex roots of $f(x)=0$ and let v denote the number of changes of sign in the sequence $\{a_i\}$. It is shown that $p+r+2q\leqq v$. The fact that the number of positive roots of an algebraic equation with real coefficients, which has only real roots, is equal to the number of changes of sign does not extend to the present case, even if all the roots are positive. This is shown by $f(x)=(y^2+1)(2-y)$, where $y=x^{\frac{1}{4}}$, which, since principal values are understood, vanishes only for $x=16$. If $f(x)$ does not vanish if $\log x$ is not the principal value it is shown that $p+r+2q=n$. *O. Todd-Taussky* (London).

Comessatti, Annibale, e Conforto, Fabio. Sulla deduzione delle relazioni bilineari tra i periodi d'un corpo di funzioni abeliane. Ist. Veneto Sci. Lett. Arti. Parte II. Cl. Sci. Mat. Nat. 102, 541–549 (1943).

This note deals with a derivation of bilinear relations belonging to a Riemann matrix. Suppose that ω, λ are $p\times 2p$ complex matrices such that $\left|{\omega \atop \lambda}\right|$ has nonvanishing determinant. It is shown that $\omega M\omega_{-1}=0$ for $M=-\det(m)m^{-1}$ with $m=\omega_{-1}\lambda-\lambda_{-1}\omega$, where ω_{-1}, λ_{-1} denote the transposes of ω, λ. *O. F. G. Schilling* (Chicago, Ill.).

Soudan, Robert. Substitution linéaire dans une forme quadratique. C. R. Séances Soc. Phys. Hist. Nat. Genève 63, 71–73 (1946).

A proof of the law of inertia for matrices of the third order. *C. C. MacDuffee* (Río Piedras, P. R.).

Ore, Aadne. On doubly symmetric determinants and corresponding homogeneous linear systems of equations. Norsk Mat. Tidsskr. 24, 65–73 (1942). (Norwegian)

It is shown that a determinant D of order k which is symmetric with respect to both diagonals is the product of two symmetric determinants D' and D'' of order $k/2$ if k is even and of order $(k\pm 1)/2$ if k is odd. The characteristic vectors of a doubly symmetric matrix are shown to be symmetric or antisymmetric according as the corresponding characteristic roots belong to D' or D''. *W. Feller.*

Ore, Aadne. On the reducibility of determinants. Norsk Mat. Tidsskr. 27, 10–12 (1945). (Norwegian)

A class of matrices is constructed for which the results of the preceding review subsist except that D' and D'' are no longer necessarily symmetric. The doubly symmetric matrices are a special case. *W. Feller* (Ithaca, N. Y.).

Parodi, M. Application des propriétés de trois types de déterminants au calcul des fréquences propres de systèmes oscillants couplés. Rev. Gén. Électricité 47, 358–363 (1940).

Vaulot, A.-E. Application de trois types de déterminants au calcul des fréquences propres de systèmes oscillants couplés. Rev. Gén. Électricité 48, 352–353 (1940).

Parodi, Maurice. Sur un type de polynômes rencontré dans l'étude des filtres électriques. Rev. Gén. Électricité 51, 142–144 (1942).

In the first paper Parodi introduces the determinant $D_n(x)$ with n rows, having x on the main diagonal, -1 on each of the two adjacent diagonals, and zeros elsewhere. He also considers two other determinants which are expressible simply in terms of $D_n(x)$. He obtains recurrence formulas for $D_n(x)$, calculates its zeros and applies the results to the computation of the eigen-frequencies of some electric circuits. In the second paper Vaulot proves that

$$(*)\qquad D_n(2\cos\theta)=\sin(n+1)\theta\csc\theta.$$

In the third paper Parodi discusses $D_n(x)$ as a polynomial in x, giving a differential equation which it satisfies, expressing it as a hypergeometric function, etc. [The relation (*) can be traced at least as far back as F. J. Studnička, S.-B. Böhmisch. Ges. Wiss. Math. Nat. Cl. **1897**, no. 1; cf. E. Pascal, Die Determinanten, Leipzig, 1900, p. 156. It has been rediscovered many times.] *R. P. Boas, Jr.*

Boaga, C. Su talune relazioni ricorrenti fra le matrici normale angolari. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **1**, 389–394 (1946).

The author studies the determinant $D_n(3)$ [in the notation of the preceding review] with a view to applications in geodesy. *R. P. Boas, Jr.* (Providence, R. I.).

Paternò, Gaetano. Determinanti θ-involutori. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) **4(73)**, 3–9 (1940).

A determinant $\Delta=|a_{ik}|$, $i, k=1, 2, \cdots, n$, is called θ-involutory provided $\sum_{j=1}^{n} a_{ij}a_{ji}=\sin^2\theta$, $\sum_{j=1}^{n} a_{ij}a_{jk}=\cos^2\theta$, $i\neq k$. The author shows that

$$\Delta^2=(-1)^{n-1}\cos^{n-1}2\theta\{1+(n-2)\cos^2\theta\}.$$

Letting $f(-x)$ denote the determinant obtained from a θ-involutory determinant by subtracting x from each element of the principal diagonal, it is seen that

$$f(x)f(-x)=(-1)^n(x^2+\cos 2\theta)^{n-1}\{x^2-(n-2)\cos^2\theta-1\},$$

and hence the characteristic values of a θ-involutory matrix are found among the numbers

$$\pm(-\cos 2\theta)^{\frac{1}{2}}, \quad \pm\{(n-2)\cos^2\theta+1\}^{\frac{1}{2}}.$$

L. M. Blumenthal (Columbia, Mo.).

Tenca, Luigi. Relazioni fra i minori di matrici di Hankel. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) **8(77)**, 141–148 (1944).

The author gives several relations between the minors of a Hankel matrix which are extensions or generalizations of known results. *N. H. McCoy* (Northampton, Mass.).

Tenca, Luigi. Ricerche sui determinanti di differenze. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) **7(76)**, 127–134 (1943).

The author generalizes slightly a theorem of Raimondi [Giorn. Mat. **26**, 185–188 (1888)] to the effect that if $a_1, a_2, a_3, \cdots$ is a sequence of numbers whose nth differences are constant, equal to c, then the determinant of any $n+1$ consecutive columns of the array whose $n+1$ rows are the nth, $(n-1)$th, $\cdots$, 0th differences has the value c^{n+1}.

L. M. Blumenthal (Columbia, Mo.).

Calapaj, Giovanni. Sulle matrici permutabili con una circolante di tipo ω data. II. Boll. Accad. Gioenia Sci. Nat. Catania (3) **14**, 16–25 (1940).

If $A_0, \cdots, A_{n-1}$ are square matrices of order $p\geqq 1$ with complex elements and $\omega\neq 0$,

$$H=\left\|\begin{matrix} A_0 & A_1 & \cdots & A_{n-1}\\ \omega A_{n-1} & A_0 & \cdots & A_{n-2}\\ & \cdots & \cdots & \\ \omega A_1 & \omega A_2 & \cdots & A_0\end{matrix}\right\|$$

is called a circulant of type ω. The author considers the properties of matrices commutative with H subject to various conditions. *C. C. MacDuffee* (Río Piedras, P. R.).

Abstract Algebra

MacLane, Saunders. Some recent advances in algebra. Revista Mat. Hisp.-Amer. (4) **6**, 191–216 (1946). (Spanish)

Translation of an article in Amer. Math. Monthly **46**, 3–19 (1939).

Prenowitz, Walter. Partially ordered fields and geometries. Amer. Math. Monthly **53**, 439–449 (1946).

A betweenness relation $(a\,b\,c)$, read "b is between a and c," can be expressed in terms of a simple binary ordering $a<b$ if the comparability postulate is valid: (B5) if a, b, c are distinct, one of the relations $(a\,b\,c)$, $(b\,c\,a)$, $(c\,a\,b)$ holds. The partial ordering given by a relation $(a\,b\,c)$ in which (B5) is not valid is not in general equivalent to a binary partial ordering. As an example of this the author cites the betweenness relation for points of the real Euclidean plane.

A field is partially ordered in terms of this ternary relation if it satisfies the invariances: (I1) $(a\,b\,c)$ implies $(a+x\;b+x\;c+x)$ and (I2) $(a\,b\,c)$ implies $(ax\;bx\;cx)$ provided $x\neq 0$. It is shown that in a partially ordered field the ordering may be expressed entirely in terms of elements between 0 and 1 by the theorem: $(a\,c\,b)$ is equivalent to the existence of elements λ, μ satisfying $c=\lambda a+\mu b$, $\lambda+\mu=1$, $(0\,\lambda\,1)$, $(0\,\mu\,1)$. Every partially ordered field is of characteristic infinity. Also the only partial ordering of the rational field is the natural ordering.

An element x is said to be linearly related to a, b if there is a sequence $x_1=a$, $x_2=b$, $\cdots$, $x_n=x$ such that each x_i, $i=3, \cdots, n$, is comparable with two earlier x's. Defining a line ab as the set of x's linearly related to a, b, the author proceeds to a definition of a partially ordered geometry, using what are essentially the Hilbert axioms of order. (G1) (closure). If $a\neq b$, there exist x, y satisfying $(a\,x\,b)$, $(a\,b\,y)$. (G2) (triangle). If a, b, c do not colline and $(b\,c\,x)$, $(c\,y\,a)$, then z exists satisfying $(a\,z\,b)$, $(x\,y\,z)$. (G3) (linearity). There exists not more than one line containing a and $b\neq a$. The author introduces a partial ordering for the field of rational functions over the real field. Using this partially ordered field for coordinates, he constructs a partially ordered geometry in which the lines are partially ordered, but not simply ordered. *M. Hall, Jr.* (Columbus, Ohio).

Lesieur, Léonce. Anneaux réguliers, avec ou sans diviseurs de zéro. C. R. Acad. Sci. Paris **223**, 1083–1085 (1946).

Ore [Ann. of Math. (2) **32**, 463–477 (1931)] showed that the usual solution of linear equations is valid in "regular" rings, i.e., rings without divisors of zero in which any two elements have a common right multiple. The author introduces a modified definition of regularity for rings with divisors of zero. The ring of matrices over a regular ring is then regular. Theorems are stated on the solution of linear equations and on dependence of vectors. These results are all deduced from a general theorem on reduction of a matrix to triangular form. No proofs are given. *I. Kaplansky.*

Loonstra, F. La structure des pseudo-évaluations d'un anneau élémentaire. Nederl. Akad. Wetensch., Proc. **49**, 899–904 = Indagationes Math. **8**, 558–563 (1946).

In a commutative ring a pseudovaluation w_1 is said to precede a pseudovaluation w_2 if every w_2-nullsequence is a w_1-nullsequence. If two pseudovaluations are identified when each precedes the other, a partially ordered set is thereby obtained. Making the assumption that each ele-

ment of this set is preceded by only a finite number of elements, the author states that the set is a distributive lattice. The proof of distributivity is incorrect and, in fact, this part of the statement is false. *I. S. Cohen.*

Kolchin, E. R. Algebraic matric groups. Proc. Nat. Acad. Sci. U. S. A. **32**, 306–308 (1946).

An algebraic matric group is a group $\mathfrak{G}$ of matrices with coefficients in an algebraically closed field $\mathfrak{C}$ which is such that the condition for a nonsingular matrix α to belong to $\mathfrak{G}$ can be expressed by algebraic conditions on the coefficients of α. If the group $\mathfrak{G}$ is solvable (in the sense that it contains a normal chain of algebraic subgroups whose factor groups are Abelian) and connected (i.e., irreducible as an algebraic variety), then $\mathfrak{G}$ is proved to be reducible to triangular form (this is a generalization of Lie's theorem for solvable Lie groups; it is remarkable that the result holds even in the case of fields of characteristic $p>0$, while the corresponding result for Lie algebras is not always true when the characteristic is not 0). An algebraic matric group $\mathfrak{G}$ is called anticompact if $\mathfrak{G}$ does not contain any element not equal to E (the unit matrix) which is of finite order not divisible by the characteristic of $\mathfrak{C}$. If $\mathfrak{G}$ is solvable, then it is anticompact if and only if it is reducible to special triangular form (i.e., zeros below the main diagonal, ones above the main diagonal); this means, in particular, that $\mathfrak{G}$ is then of rank zero (or nilpotent) in the classical terminology. An algebraic matric group $\mathfrak{G}$ is called quasicompact if it has no nontrivial anticompact algebraic subgroup. If $\mathfrak{G}$ is reducible to triangular form and is quasicompact, then $\mathfrak{G}$ is reducible to diagonal form (and is therefore Abelian).

C. Chevalley (Princeton, N. J.).

Kolchin, E. R. The Picard-Vessiot theory of homogeneous linear ordinary differential equations. Proc. Nat. Acad. Sci. U. S. A. **32**, 308–311 (1946).

The object of the paper is to indicate how it is possible to give a rigorous exposition of the Picard-Vessiot theory of homogeneous linear differential equations, using the algebraic notions introduced by Ritt in the theory of differential equations and the theory of algebraic matric groups as developed by the author [see the preceding review]. Let $\mathfrak{F}$ be a differential field over an algebraically closed field of constants $\mathfrak{C}$ of characteristic 0, and let $L(y)$ be a linear differential polynomial of the nth order with coefficients in $\mathfrak{F}$. By adjoining to $\mathfrak{F}$ n linearly independent solutions of the equation $L(y)=0$, one obtains a differential field $\mathfrak{G}$; the automorphisms of $\mathfrak{G}/\mathfrak{F}$ form a group which can be represented as an algebraic group $\mathfrak{G}$ of $n\times n$ matrices with coefficients in $\mathfrak{C}$. The Galois theory of $\mathfrak{G}$ with respect to $\mathfrak{F}$ is then considered; it turns out that the intermediary differential fields between $\mathfrak{F}$ and $\mathfrak{G}$ correspond in a one-to-one way to the algebraic subgroups of $\mathfrak{G}$. Besides giving a rigorous treatment of the theory, the methods of the author make it possible to sharpen the main theorem, which refers to the case where $\mathfrak{G}$ (or at least the component of the identity in $\mathfrak{G}$) is solvable. In this case, the equation $L(y)=0$ is solvable by a combination of the following operations: taking integrals, exponentials of integrals and solving algebraic equations. If one does not allow all of these types of operation but only some of them, one arrives at various classes of differential equations, and the author characterizes these classes by properties of their groups (such as solvability, anticompactness or quasicompactness either for $\mathfrak{G}$ or for the component of the identity in $\mathfrak{G}$).

C. Chevalley (Princeton, N. J.).

Barinaga, J. Introduction to Henselian arithmetic. Euclides, Madrid **1**, 129–160 (1941). (Spanish)

Shafarevich, I. R. On Galois groups of $\mathfrak{p}$-adic fields. C. R. (Doklady) Acad. Sci. URSS (N.S.) **53**, 15–16 (1946).

Let $K\supset k\supset R$, where R is a $\mathfrak{p}$-adic completion of the rational field, k/R is normal with Galois group $\mathfrak{F}=\{\sigma,\tau,\cdots\}$ and K/k is Abelian with group $\mathfrak{A}=\{\alpha,\beta,\cdots\}$. Let K be class field to the group H of numbers in k and assume H is invariant under $\mathfrak{F}$ so that K is normal over R. Then the Galois group $\mathfrak{G}$ of K/R is a Schreier group extension [Zassenhaus, Lehrbuch der Gruppentheorie, Teubner, Berlin, 1937, p. 89] such that $\mathfrak{G}/\mathfrak{A}\cong\mathfrak{F}$; $\mathfrak{G}$ is completely determined by the automorphisms α^σ and a factor set $\alpha_{\sigma,\tau}$ in $\mathfrak{A}$. The author proves the following formula. Let A be the simple algebra with center R, degree $(K:k)=m$ and invariant $1/m$. Then $A=(k/R, a_{\sigma,\tau})$ for some factor set $a_{\sigma,\tau}$ of numbers of k. The norm residue symbol $\alpha(a)=(a, A/p)^{-1}$ maps nonzero elements $a\epsilon k$ onto elements $\alpha(a)\epsilon\mathfrak{A}$. Then $(\alpha(a))^\sigma=\alpha(a^\sigma)$ and $\alpha_{\sigma,\tau}=\alpha(a_{\sigma,\tau})$. [For terminology and existence theorems see Deuring, Algebren, Ergebnisse der Math., v. 4, no. 1, Springer, Berlin, 1935, chaps. V, VII.] The proof uses Nakayama's formula on the relation between factor sets and norm groups [Math. Ann. **112**, 85–91 (1935)], followed by computation with factor sets. Nakayama has recently applied similar computations to local class field theory [Jap. J. Math. **18**, 877–885 (1943); these Rev. **7**, 363]. *G. Whaples* (Madison, Wis.).

★**Spampinato, Nicolò. Caratterizzazione delle funzioni di variabile ipercomplessa analitiche secondo Ringleb fra le funzioni a derivata caratteristica.** Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 91–95. Edizioni Cremonense, Rome, 1942.

The author's principal result may be stated as follows. Let A be an algebra over the reals, B the enveloping algebra of its right and left multiplications. Then a function in A is analytic in the sense of Ringleb if and only if the matrix of its partial derivatives lies in B. [For references and further details on this point of view, cf. J. A. Ward, Duke Math. J. **7**, 233–248 (1940); these Rev. **2**, 122]. From this it follows that these functions coincide with the author's "totally derivable" functions if A is commutative. Finally it is shown that they coincide with the author's functions with a "characteristic derivative" if and only if A is simple.

I. Kaplansky (Chicago, Ill.).

Spampinato, Nicolò. Sulle funzioni in un'algebra complessa dotata di modulo. Applicazioni alle algebre dei ternioni, dei numeri triduali e dei numeri tripotenziali. Univ. Roma e Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **2**, 193–231 (1941).

Let A be an algebra with unit element over the complex numbers. Recapitulating earlier work [Atti Acad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) **21**, 626–631 (1935)], the author defines functions in A with a "unique characteristic left derivative" and identifies them with functions which are "totally left differentiable." [Cf. also J. A. Ward, Duke Math. J. **7**, 233–248 (1940); these Rev. **2**, 122]. A function totally differentiable at x_0 admits an expansion

in powers of $x-x_0$. Conversely a power series in $x-x_0$ is totally differentiable at x_0, and in the commutative case it is also differentiable at every point in the region of convergence. In the noncommutative case this is false; power series for which it does hold are called "special" by the author. For the various algebras of order three the author gives detailed illustrations of these concepts. A final purely algebraic section deals with the number of roots of polynomial equations with coefficients in these algebras of order three. [Cf. the following review.]

I. Kaplansky (Chicago, Ill.).

★Spampinato, Nicolò. Il teorema fondamentale dell'algebra per una qualunque algebra complessa dotata di modulo. Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 96–104. Edizioni Cremonense, Rome, 1942.

Let A be an algebra with unit over the complex numbers; suppose it has order s and that its radical has order r. Let $f(x)$ be a general polynomial of degree n, i.e., a sum of terms $a_0xa_1x\cdots xa_k$ ($a_i\epsilon A$, $k=0,\cdots,n$). The paper is devoted to a proof that $f(x)=0$ has n^{s-r} roots. The sense in which the term general ("generica") is to be taken is not made precise.

I. Kaplansky (Chicago, Ill.).

THEORY OF GROUPS

Niggli, Paul. Neuformulierung der Kristallographie. Experientia 2, 336–349 (1946).

Um die Kristallklassen zu charakterisieren, verwendete man bisher eine im mathematischen Sinne als abstrakt zu bezeichnende Symbolik der Symmetrieelemente, etwa die von A. Schoenfliess eingeführte. Naturgemäss gibt sie uns keinen Einblick in den Aufbau der Gruppen, insbesondere lassen sich die Untergruppen nicht daraus ablesen. In der reinen Mathematik geht man daher zu einer Darstellung der Gruppen durch lineare Substitutionen über. Für viele Zwecke der Kristallographie und insbesondere der Stereochemie erweist sich aber ein etwas anderer Weg als vorteilhafter. Dies wird in vorliegender Abhandlung auseinandergesetzt. Nach G. Pólya [Acta Math. 68, 145–253 (1937)] ordnet man in einer Gruppe der Ordnung n jedem Element m-ten Grades das Symbol f_m^i zu, wo i der Index ist, $m\cdot i=n$. Man erhält dadurch als "Formel" für die Kristallklasse einen Ausdruck der Form $n^{-1}\sum cf_m^i$. In ihm fasst man geometrisch gleichartige Elemente passend durch Klammern zusammen. Um sich der Darstellungstheorie noch einen Schritt zu nähern und etwa die verschiedenartigen Elemente der Ordnung zwei voneinander zu unterscheiden, verfeinert der Verfasser diese Schreibweise durch Einführung von Symbolen für die Drehungen um $2\pi/k$, die Drehinversionen und die Spiegelinversionen. Die sich hierdurch ergebenden Formeln für die einzelnen Symmetrieelemente und für die Kristallklassen werden eingehend auseinandergesetzt. Sodann wird gezeigt, welche Probleme der Kristallographie hiermit gelöst werden können. Insbesondere ergeben sich die Formeln für die Untergruppen aus denen für die ganze Gruppe durch einen einfachen Rechnungsprozess, "sodass sich mathematisch alle Aufgaben der Formvieldeutigkeiten, der Formenänderung bei Abbau der Symmetrieelemente und der Isomerie spielend lösen lassen."

J. J. Burckhardt (Zürich).

Mignosi, Giuseppe. Sulla definizione di gruppo. Univ. Roma e Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) 4, 60–73 (1943).

A given group may be defined abstractly, and also as a group of one-to-one mappings of a set into itself. The author discusses the definitions of each of these types proposed by various authors and selects the one of each kind which seems most satisfactory. He then proves the equivalence of these two definitions. The method of proof and the result itself are well known.

S. A. Jennings.

Casadio, Giuseppina. Costruzione di gruppi come prodotto di sottogruppi permutabili. Univ. Roma e Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) 2, 348–360 (1941).

The following problem has been suggested by Ore [Duke Math. J. 3, 149–174 (1937)]. Given two groups A, B, find all groups G which are the union of permutable subgroups $\bar{A}$, $\bar{B}$ such that $\bar{A}\cong A$, $\bar{B}\cong B$. The present paper is concerned with a special case of this problem, namely when A and B contain normal subgroups A', B' both isomorphic to a given group C. Under these circumstances all groups G are determined such that G is the union of permutable subgroups $\bar{A}$ and $\bar{B}$ whose intersection $\bar{C}$ is normal in both $\bar{A}$ and $\bar{B}$, where $\bar{A}\cong A$, $\bar{B}\cong B$, $\bar{C}\cong C$. The groups G are determined in terms of factor systems and automorphisms satisfying conditions which are too complicated to be reproduced here. The case when $C=1$ has been considered earlier by Zappa.

S. A. Jennings (Vancouver, B. C.).

Guha, U. On the endomorphic mappings $\{m\}$ of a group. Bull. Calcutta Math. Soc. 38, 101–107 (1946).

If G is a group, then $M=M(G)$ is the smallest positive integer such that a homomorphism of G upon an Abelian subgroup, called $Z=Z(G)$, is obtained by mapping x in G upon x^M [see F. W. Levi, J. Indian Math. Soc. (N.S.) 8, 1–9 (1944); these Rev. 6, 40]. Mapping x upon x^n is shown to effect an automorphism of G if, and only if, it induces an automorphism in Z and $n=1+kM$; and $1+kM$ and $1+k'M$ induce the same automorphism in G if, and only if, $z^k=z^{k'}$ for every z in Z. The group G is an L-group if its commutators obey the associative law: $(a,(b,c))=((a,b),c)$ [see F. W. Levi, J. Indian Math. Soc. (N.S.) 6, 87–97 (1942); these Rev. 4, 133]. The positive integer m is the invariant $M(G)$ of some L-group G if, and only if, $M\not\equiv 2$ modulo 4; and mapping x onto x^i effects an endomorphism of the L-group G if, and only if, $i\equiv i^2$ modulo $M(G)$.

R. Baer (Urbana, Ill.).

Kaloujnine, Léo. Sur les p-groupes de Sylow du groupe symétrique du degré p^m. (Suite centrale ascendante et descendante.) C. R. Acad. Sci. Paris 223, 703–705 (1946).

In two previous notes [same C. R. 221, 222–224 (1945); 222, 1424–1425 (1946); these Rev. 7, 239; 8, 13] the author showed that the Sylow subgroups P_m of the symmetric group of degree p^m may be represented as permutations of m-dimensional vectors with components in $GF(p)$, and studied their structure for the case $m=2$. In the present note the upper and lower central series of P_m are studied. These series are shown to be identical and of length $c=p^{m-1}$, the factor groups of the series are shown to be of type $(1,1,\cdots,1)$ and a formula is given for their order. The proofs are by induction on m.

D. G. Duncan.

Wiman, A. Über mit Diedergruppen verwandte p-Gruppen. Ark. Mat. Astr. Fys. 33A, no. 6, 12 pp. (1946).

Let l be the class of a group, i.e., the number of terms in its upper central series. Then for $l=1$ the group is Abelian,

The paper by Zappa mentioned at the end of the review is reviewed on p. 367 of V. 8.

and in general $l \leqq n-1$. This note determines and classifies into three types all p-groups of the maximal class $n-1$ having an Abelian subgroup A of order p^{n-1}. The classification corresponds to that for $p=2$ in which (in a slightly different notation from the author's) two generators a and b satisfy $a^{2^{n-2}}=-1$, $b^2=(-1)^\alpha$, $b^{-1}ab=(-1)^\beta a^{-1}$, and where for type I (dihedral) $\alpha=\beta=0$; for type II (dicyclic) $\alpha=1$, $\beta=0$; and for type III $\alpha=0$, $\beta=1$. The corresponding classification for $p>2$ is arrived at by taking two generators s_1 (for a, in the Abelian subgroup A) and s (for b, not in A) and defining successively s_{i+1} to be the commutator $s_i^{-1}s^{-1}s_is$, for $i=2, \cdots, n-2$. The element s_{n-1} in the center becomes -1 for $p=2$, and in general $s^p=s_{n-1}^\alpha$, so that s is of order p or p^2. The commutator $K_i=s_i^{-1}s^{-p}s_is^p$ is expanded in the form $K_i=\prod_{r=1}^p s_{i+r}^{c(r)}$, where $c(r)=p!/r!(p-r)!$, and we have $K_i=1$ for $i=1, 2, \cdots, n-2$. The correspondingly defined K_0 is of the form s_{n-1}^β. The three types depend as before on whether or not $\alpha\equiv 0$ and $\beta\equiv 0 \pmod p$. For each of the first two types ($\beta\equiv 0$) there is a unique abstract p-group of given order p^n ($n\geqq 3$). However, in type III (for $n\geqq 4$) there are $\mu\geqq 1$ distinct groups, where $\mu=(n-2, p-1)$. The author discusses some classifications of p-groups by Burnside, Séguier, Potron, P. Hall, and Easterfield, and makes the conjecture that all groups of order p^n and class $n-1$ should lie one each in certain families, defined in a natural way for values of p and n exceeding a certain family minimum, the minimum for p being perhaps in general equal to $n-2$.

J. S. Frame (East Lansing, Mich.).

Koulikoff, L. On the theory of Abelian groups of arbitrary power. Rec. Math. [Mat. Sbornik] N.S. **16(58)**, 129–162 (1945). (Russian. English summary) [MF 13007]

This is a continuation of an earlier paper [same Rec. N.S. **9(51)**, 165–181 (1941); these Rev. **2**, 308]. It is shown that an Abelian primary group is decomposable into a direct sum of cyclic subgroups if and only if the group is a set theoretical sum of an ascending sequence of subgroups in every one of which the heights of the elements are uniformly bounded. (An element a of a primary group A is said to have the height n in A if n is the largest number such that $a\epsilon p^nA$.) This theorem contains former results of Prüfer and contains also the new result that every subgroup of an Abelian group which is a direct sum of cyclic subgroups is itself a direct sum of cyclic subgroups. Furthermore, the author introduces the notion of a basic subgroup of a primary group. All basic subgroups of a given group are isomorphic to each other and are direct sums of cyclic groups. The factor group of a primary group with respect to a basic subgroup is a complete group. (An Abelian group A is called complete if $nA=A$ for every integer n.) In conclusion the author shows that some results of H. Ulm on countable Abelian groups cannot be extended to groups of arbitrary power. *W. Hurewicz* (Cambridge, Mass.).

Kishkina, Z. Endomorphisms of p-primitive Abelian groups without torsion. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **9**, 201–232 (1945). (Russian. English summary) [MF 13328]

Let G be an Abelian group without elements of finite order and with a finite basis consisting of elements $u_1, u_2, \cdots, u_n$. Given a prime number p, the group G is called p-primitive if every element g of G satisfies a relation of the form $p^mg=k_1u_1+\cdots+k_nu_n$ (where m and k_i are integers). The paper contains an exhaustive study of the endomorphisms of p-primitive groups. An isomorphism is established between the ring of endomorphisms of a p-primitive group of rank n and a certain ring of matrices of order n over the ring R_p of rational numbers whose denominators are powers of p. Furthermore, the author establishes conditions for the existence of nontrivial endomorphisms (that is, endomorphisms which are not of the form $g\rightarrow mg$).

W. Hurewicz (Cambridge, Mass.).

Lombardo-Radice, L. Gli automoduli primitivi delle algebre legate a gruppi abeliani di ordine finito. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **1**, 686–690 (1946).

Let G be an Abelian group of order n, F a field containing the nth roots of unity and with a characteristic prime to n. The author finds the primitive idempotents of the group ring of G over F. The result is essentially known [van der Waerden, Moderne Algebra, 2d edition, v. 2, Springer, Berlin, 1940, pp. 178–180; these Rev. **2**, 120].

I. Kaplansky (Chicago, Ill.).

Zappa, Guido. Sui gruppi di Hirsch supersolubili. I. Rend. Sem. Mat. Univ. Padova **12**, 1–11 (1941).

Infinite discrete soluble groups in which the maximal condition for subgroups holds have been called S-groups by Hirsch [Proc. London Math. Soc. (2) **44**, 53–60, 336–344 (1938); **49**, 184–194 (1946); these Rev. **8**, 132], who proved that every such group G contains series of subgroups of the form $G=G_0\supset G_1\supset\cdots\supset G_l=1$, where G_i is normal in G_{i-1} and G_{i-1}/G_i is either cyclic of prime order or an infinite cyclic group, $i=1, 2, \cdots, l$. Such series are similar to composition series for finite groups, and will be called composition series of G; similarly if G contains a composition series all of whose terms are normal in G such a series will be called a chief series of G; a composition or chief series of shortest length will be called a minimal composition or chief series. In the present paper those S-groups which actually have chief series are studied. Such groups are said to be supersoluble S-groups and are natural generalisations of the finite supersoluble groups ("groups with complete principal chains") discussed previously by the author [Rend. Sem. Mat. Univ. Roma (4) **2**, 323–330 (1938)] and Ore [Duke Math. J. **5**, 431–460 (1939)]. It is shown that the derived group of a supersoluble S-group G is nilpotent and that the elements of G of odd finite order form a characteristic subgroup D. The factors in any two minimal chief series of G are proved to be isomorphic in some order and the odd prime factors in such minimal chief series are isomorphic to the factors of a chief series of the finite subgroup D.

S. A. Jennings (Vancouver, B. C.).

Zappa, Guido. Sui gruppi di Hirsch supersolubili. II. Rend. Sem. Mat. Univ. Padova **12**, 62–80 (1941).

The previous result on the factors of minimal chief series of a supersoluble S-group [cf. the preceding review] is shown to hold also for minimal composition series, and the Sylow subgroups of supersoluble S-groups are investigated. The maximal subgroups of G of odd prime power order are necessarily Sylow subgroups of the subgroup D, and hence are conjugate in G, but examples are given of supersoluble groups with maximal subgroups of orders 2^a and 2^b with $a\neq b$, so that in general such subgroups are not even isomorphic. Finally a generalisation of a theorem of Ore [loc. cit. in the preceding review, theorem 10] on finite supersoluble groups is obtained. Let H be a subgroup of K: then H is said to be of index ∞^1 in K if there exists an element $a\epsilon K$ such that $a^iH\neq H$ for all integers $i\neq 0$, while every element of K is in one of the cosets a^iH. In terms of this

definition Ore's result generalises as follows. An S-group G is supersoluble if and only if from every chain of subgroups Σ of G there may be obtained by refinement a chain Σ' such that the index of every subgroup of Σ' in the preceding subgroup is ∞^1 or a prime. *S. A. Jennings.*

Kurosh, A. Composition systems in infinite groups. Rec. Math. [Mat. Sbornik] N.S. 16(58), 59–72 (1945). (Russian. English summary) [MF 13013]

This is a development of an earlier paper by the author [Math. Ann. 111, 13–18 (1935)]. Given two normal systems of subgroups of a group G (with operators), Zassenhaus's method can be used to obtain isomorphic refinements of these two systems. Generally speaking, such refinements will be systems with repetitions. Conditions under which repetitions can be avoided are investigated. The general results are applied to the case of well-ordered normal systems. *W. Hurewicz* (Cambridge, Mass.).

Kurosh, A. The Sylow subgroups of zero-dimensional topological groups. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 9, 67–78 (1945). (Russian. English summary) [MF 12766]

An element a of a topological group G is called a p-element (p is a prime) if the sequence $a, a^p, a^{p^2}, \cdots, a^{p^n}, \cdots$ converges to the unit element. A topological group is called a topological p-group if all its elements are p-elements. A subgroup of G is called a p-subgroup if all its elements are p-elements in G. The main result is as follows. Suppose that the topological group G possesses a complete system of neighborhoods of the unit element all of which are invariant subgroups and suppose that there is a closed p-subgroup P in G such that the class of subgroups conjugate to P is bicompact. Then for every p-subgroup Q of G there exists a subgroup P_1 conjugate to P such that (P_1, Q) is a p-subgroup of G. From this theorem the author derives the further result that if a topological group satisfying the above requirement about neighborhoods of the unit element contains a bicompact class of Sylow (i.e., maximal) p-subgroups, then all the Sylow p-subgroups of G are conjugate. This result is a generalization of the well-known theorem of van Dantzig about Sylow p-subgroups in compact zero-dimensional groups. *W. Hurewicz* (Cambridge, Mass.).

Bochner, Salomon, and Montgomery, Deane. Locally compact groups of differentiable transformations. Ann. of Math. (2) 47, 639–653 (1946).

Les auteurs traitent des groupes localement compacts G, représentés par des transformations (C^2) d'une variété (C^2), qui ne possèdent pas d'ensembles ouverts de points invariants (à l'identité près). Ils démontrent que ces groupes sont nécessairement des groupes de Lie.

Les conditions de différentiabilité entraînent d'après les auteurs [mêmes Ann. (2) 46, 685–694 (1945); ces Rev. 7, 241] la continuité des dérivées (par rapport aux coördonnées) simultanément par rapport aux coördonnées et aux paramètres, et par suite la non-existence de sous-groupes arbitrairement petits [H. Cartan, Sur les groupes de transformations analytiques, Actual. Sci. Indus., no. 198, Hermann, Paris, 1935; S. Bochner, mêmes Ann. 46, 372–381 (1945); ces Rev. 7, 114]. Le procès d'intégration imaginé par les auteurs [loc. cit.] conduit à un théorème préparant la construction de sous-groupes à un paramètre, qui est effectuée par des méthodes de J. von Neumann et É. Cartan alors devenues classiques, de même que la démonstration de l'analyticité du groupe G qui y repose.

Une conséquence de ces recherches est le théorème: le groupe de toutes les homéomorphismes analytiques d'une variété complexe analytique est un groupe de Lie. *H. Freudenthal* (Amsterdam).

Oxtoby, John C. Invariant measures in groups which are not locally compact. Trans. Amer. Math. Soc. 60, 215–237 (1946).

This paper presents an investigation of the existence and properties of invariant measures in complete metric groups which are not locally compact. The discussion is illuminated by many examples. A Borel measure is defined to be a countably-additive nonnegative set function defined for all (and only for) Borel sets, which may assume the value infinity, but is required to be finite and positive for at least one set and zero for points. It is first shown that, given a left-invariant Borel measure in a non-locally-compact complete separable metric group, any neighborhood contains noncountably many disjoint congruent compact sets whose common measure is finite and positive. The proof, previously unpublished, is due to Ulam. Using a more intricate construction the author shows that these compact sets can be so chosen that their sum is compact. The Haar measure in a locally compact separable metric group is an invariant Borel measure which has additional properties. For instance, some open sets have finite measure, every compact set has finite measure, and every measurable set is the countable sum of sets of finite measure. The above theorems show that all of these additional properties fail if the group is not locally compact. The author next constructs a left-invariant Borel measure in any complete separable metric group G which is dense in itself. To do this he remetrizes with a left-invariant metric and then constructs a Cantor set C which, for every n, is the sum of 2^n disjoint congruent compact sets whose common diameter is less than the smallest distance between them. An invariant Carathéodory outer measure is introduced in G by means of coverings in such a way that the measure of each of the above 2^n compact sets is 2^{-n}. The set C can be varied to yield an infinite increasing sequence of distinct invariant Borel measures for G. Any such measure may be the Haar measure for G in some locally compact topology, but the infinite product group G_1 of the additive group of integers with itself is an example of a group in which the constructed measure is never the Haar measure. The author next exhibits maximal and minimal extensions to the whole of a metric group G of an invariant measure defined on a subset. In the above example the measure constructed on G_1 is neither the maximal nor the minimal extension of its values on C. The measure derived by restricting a general measure m to the Borel field generated by the sets of finite m measure is called the σ-finite contraction of m; m is called σ-finite if it is its own σ-finite contraction. Haar measure in a locally compact group is generally taken to be σ-finite; this is necessary for uniqueness since the maximum and minimum extensions of Haar measure are not ordinarily the same. A left-invariant Borel measure in a group G is called a Haar-Borel measure if its σ-finite contraction is equal to the Haar measure in G for some locally compact topology in G. It is shown that, if a metric group G contains a locally compact normal subgroup H, dense in itself, then there exists a Haar measure in G which is an extension of Haar's measure in H. In order better to understand Haar and Haar-Borel measures the author considers more general measures, called W

and Weil-Borel measures. It is shown that the measure constructed in the above group G_1 is not a Weil-Borel measure, and, in fact, that this group is one of a class of topological groups in which no Weil-Borel measures can exist. The paper concludes with several pages of examples.

L. H. Loomis (Cambridge, Mass.).

Dubreil, P., et Dubreil-Jacotin, M.-L. Équivalences et opérations. Ann. Univ. Lyon. Sect. A. (3) **3**, 7–23 (1940).

The subject of this paper is "regular equivalences," which are equivalence relations in a groupoid such that $a\sim b$ implies $ax\sim bx$ and $xa\sim xb$ for all x. If one multiplies classes of equivalent elements in the natural way a new groupoid is defined, called the "set quotient." In fact, regularity is necessary and sufficient for the multiplication to be single valued. An "algebra," following Birkhoff [Proc. Cambridge Philos. Soc. **31**, 433–454 (1935)], is a set which is a groupoid with respect to each of several operations. An equivalence in an algebra is "homomorphic" if it is regular with respect to each operation. In a group a regular equivalence is implied by and implies the existence of a normal subgroup. In a ring a homomorphic equivalence exists for each ideal.

For an Abelian semi-group (associative groupoid with the cancellation law) the authors investigate the conditions that the set quotient shall be a group. Two such are found which are both necessary and sufficient. They are these: the class U which is the identity of the quotient group is a subgroupoid, such that (A) if ϵ is in U and $\alpha\epsilon$ is in U then α is in U; (B) for each ξ there exists at least one η such that $\xi\eta$ is in U. Furthermore, they show that the equivalence can be expressed in terms of U: $\xi\sim\xi'$ if and only if $\xi\epsilon'=\xi'\epsilon$ with ϵ' and ϵ in U.

Several standard examples are shown to be special cases of this process, especially that of completing a domain of integrity E. This amounts to constructing first the direct square E^2, and then the quotient group E^2/U, where U is the subgroupoid of pairs (ϵ, ϵ). The details of this are carried out for completing the natural numbers to the rational integers and for completing the rational numbers to the reals.

H. Campaigne (Arlington, Va.).

ANALYSIS

Theory of Sets, Theory of Functions of Real Variables

***Denjoy, Arnaud. L'Énumération Transfinie. Livre I. La Notion de Rang.** Gauthier-Villars, Paris, 1946. xxxvii+206 pp.

This is the first half of a treatise on the transfinite numbers of the second class. It contains a detailed index of the second half which is to include chapter IV (Definition of numbers of the second class by sums of special permutations), chapter V (Canonical sequences) and two notes on controversial matters. There is also a fairly comprehensive bibliography (compiled by G. Choquet) of work in this field in 1920–1946. The main characteristics of this work, in addition to the actually new results [some of which have been announced in C. R. Acad. Sci. Paris **212**, 885–888; **213**, 430–433 (1941); **221**, 429–432, 679–682 (1945); **222**, 981–983 (1946); these Rev. **3**, 73; **5**, 113; **7**, 194, 419] are a meticulous choice of language (as was to be expected) and careful analyses of the proofs of familiar results.

In chapter I (The ordering of classes) the basic ideas of the theory of series (ordered classes) are introduced. Considerable use is made of the concept of regularity. A linear set is said to be regular in an interval $(\alpha\beta)$ if it contains all its one-sided (geometrical) limit points which lie in the interior of $(\alpha\beta)$. It can be shown that a series E, which contains a countable subseries Δ such that between any two elements of E which are not both in Δ there is at least one element of Δ, is (ordinally) similar to a regular linear set. From this can be obtained a generalisation of the Cantor-Baire theorem on well-ordered contracting series of closed sets: if a monotone family φ of closed sets F is such that all members of φ (properly) included in a particular $F\epsilon\varphi$ are actually included in some $F_1\subset F$, $F_1\epsilon\varphi$, then φ is countable.

In the section on well-ordered series we find a result which enables transfinite ordinals to be avoided in the proofs of certain theorems [cf. K. Kuratowski, Fund. Math. **3**, 76–108 (1922)]. Consider a monotone family φ of sets which is closed under multiplication and contains a maximal set H. Suppose that with each $G\epsilon\varphi$ there is associated a set $\Lambda(G)\subset G$ in such a way that (i) there is at most one $L\epsilon\varphi$ such that $\Lambda(L)=L$ (then L is the minimal set of φ), (ii) if $\Lambda(G)$ is the empty set then every member of φ different from G includes G while if $\Lambda(G)$ is not empty then it belongs to φ and every set in φ, different from $\Lambda(G)$ and G, either contains G or is contained in $\Lambda(G)$. Then such a family is well-ordered by the relation of inclusion and is completely determined by a knowledge of the function Λ and the set H.

In chapter II (Ordinal numbers) the author insists on the distinction between three ideas: (i) that of the rank of an element a_0 in a (well-ordered) series E_0, (ii) that of ordinal number and (iii) that of the order-type of a well-ordered series. These distinctions were not made by Cantor or his followers. Consider all well-ordered series E containing an element a such that the initial segment of E determined by a is (ordinally) similar to the initial segment of E_0 determined by a_0. All such elements a, disregarding their nature and the behaviour of their successors, have in common a character which is their rank. This is invariant under (ordinal) similarities, is characteristic for the element in question and depends only on the initial segment determined by it. The concept of rank applies to certain series which are not well-ordered; this extension is discussed in some detail. An ordinal number is an element of a certain well-ordered series, which is a standard scale for rank, in fact the series $1, 2, \cdots, \omega, \cdots$. The ordinals are used as labels to indicate the rank of elements in well-ordered series. The order-type (or type of ordering) of a series is defined as usual. Order-types can be well-ordered and can therefore be enumerated by the ordinal numbers and indeed if τ_α corresponds to the ordinal α then for $\alpha=n$ (finite) τ_α is the order-type of $[1, 2, \cdots, n]$ (where $n=\alpha$ is included) while for α transfinite τ_α is the order-type of $[1, 2, \cdots, \alpha)$ (where α is not included). The idea of order-type appears to be unnecessary in the theory of functions but it is unavoidable when we consider the addition and multiplication of ordinal numbers. Some account of the arithmetic of ordinals is given; the definition of $\alpha+\beta$ always coincides with that given by Cantor but that for $\alpha\times\beta$ only coincides with Cantor's when β is finite.

Chapter III is concerned with the numerical and geometrical representations of the ordering of countable sets. The problem of re-ordering, in all possible ways, a given

enumerated countable set is equivalent to the problem of determining all possible permutations of the set N of all positive integers. This is equivalent to (i) the formation of a particular permutation representative of each order-type and (ii) the formation of all elementary permutations, i.e., those of type ω. The representation of a countable order-type implicit in the work of Lebesgue [J. Math. Pures Appl. (6) 1, 139–216 (1905), in particular, p. 213] is not satisfactory for the study of these problems. Even after choosing a fixed enumeration of the rational numbers there is a continuous infinity of real numbers associated with a given permutation. Several alternatives are discussed which are in general (1, 1). For instance, we first represent a number $x\epsilon[0, 1]$ in the form $x=\sum(a_n/n!)$, where $0\leqq a_n<n-1$. With the sequence $\{a_n\}$ we can associate a permutation of N. For if we have fixed the relative order of $1, 2, \cdots, n-1$ (by use of $a_1, a_2, \cdots, a_{n-1}$) we can use the fact that a_n has n possible values to place n relative to $1, 2, \cdots, n-1$, i.e., before the first of these, after the last or in the $n-2$ gaps between consecutive elements. Suitable conventions lead to interesting developments, for instance, conditions on $\{a_n\}$ necessary and sufficient in order that to it (or to x) there should correspond a well-ordered permutation. These conventions can be illustrated by sieves.

The author introduces a geometrical representation of a permutation of N which serves to illuminate the numerical representations and to facilitate the study of certain problems on permutations. This model is too complicated to describe here. *J. Todd* (London).

Walker, A. G. A theorem on ordered sets. J. London Math. Soc. **21**, 9–10 (1946).

The author ~~states~~ proves the following theorem. Let X be a simply ordered set of elements x which has a first element x_0 and is such that (A) every set of nonoverlapping intervals in X is denumerable; (B) every bounded ascending progression in X has a limit; (C) every x (except the last element if it exists) is the limit of a descending progression $y_n(x)$ such that, for all n, $x_1<x_2$ implies $y_n(x_1)\leqq y_n(x_2)$. Then X contains a denumerable dense subset.

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~~The proof appears to be incorrect. Let X be the closed interval $[0, 2]$ in the real number system and let $y_n(x)=x+2^{-n}(1-x)$ for $x<1$, $y_n(x)=2^{-n}(2-x)$ for $x\geqq1$. The set X with y_n as defined satisfies the conditions of the theorem, but does not permit the carrying out of an essential step in the proof, as described on line 10 of page 10.~~

E. Hewitt (Bryn Mawr, Pa.).

Leja, F. Sur les suites de polynômes et la fonction de Green généralisée. I. Ann. Soc. Polon. Math. **18**, 4–11 (1945).

Let E be a closed and bounded set, $D(E)$ the component of the complement of E containing the point at infinity, F the boundary of $D(E)$. Let z_0 be a point of F with the following property: if $V(z_0)$ is an arbitrary closed neighborhood of z_0, the transfinite diameter of $F\cdot V(z_0)$ is positive. Let P be the set of such z_0. Then the transfinite diameter of F coincides with that of P. Other theorems involving this set P are also proved. *G. Szegö.*

Sierpiński, Waclaw. Sur la congruence des ensembles de points et ses généralisations. Comment. Math. Helv. **19**, 215–226 (1947).

Lecture at the University of Zurich.

Hadwiger, H. Ein Translationssatz für Mengen positiven Masses. Portugaliae Math. **5**, 143–144 (1946).

Dans cette note est établie la généralisation suivante d'une proposition de Steinhaus et Rademacher [Fund. Math. 1, 93–104 (1920)]: A étant un ensemble de mesure positive $m(A)$ de l'espace euclidien E_n, pour tout nombre naturel $k>1$ et tout $\epsilon>0$ il existe un $\Delta>0$ tel que, pour toutes les translations T_ν $(\nu=1, \cdots, k)$ de distances mutuelles inférieures à Δ, la mesure de l'intersection $A_1\cdot A_2\cdot \cdots \cdot A_k$ $(A_\nu=T_\nu(A))$ soit supérieure à $(1-\epsilon)m(A)$. La démonstration repose sur une inégalité très simple relative à la mesure de l'intersection d'ensembles mesurables inclus dans un même ensemblé mesurable, appliquée au Δ-voisinage d'une approximation intérieure fermée A_0 de A et aux ensembles $A_\nu^0=T_\nu(A_0)$. *C. Pauc* (Marseille).

Hadwiger, H. Eine Erweiterung eines Theorems von Steinhaus-Rademacher. Comment. Math. Helv. **19**, 236–239 (1947).

The author proves, among other things, that every set of positive measure in R_n contains a set similar to every finite set of R_n. *P. Erdös* (Syracuse, N. Y.).

Buck, R. Creighton. The measure theoretic approach to density. Amer. J. Math. **68**, 560–580 (1946).

Let $\mathfrak{D}_0$ be the Boolean algebra of subsets of the set of positive integers generated by the arithmetic progressions and the finite sets and let $\Delta(A)$ for $A\epsilon\mathfrak{D}_0$ be the density of A. If $\mu(S)=\inf\{\Delta(A):S\subset A\epsilon\mathfrak{D}_0\}$ is the outer measure induced by Δ, and $\mathfrak{D}_\mu$ is the class of all sets measurable with respect to μ in the sense of Carathéodory, then $\mathfrak{D}_\mu$ is a Boolean algebra and μ is an extension of Δ to a (finitely additive) measure on $\mathfrak{D}_\mu$. By considering some special sets of integers (e.g., primes and squares) the author proves the existence of infinite sets of measure zero and hence that $\mathfrak{D}_0\neq\mathfrak{D}_\mu$. If Z is a set of measure zero and $A\epsilon\mathfrak{D}_0$ then the symmetric difference of A and Z belongs to $\mathfrak{D}_\mu$. The author proves that not every set in $\mathfrak{D}_\mu$ is of this form, by observing that for sets of this form the value of μ is rational, whereas the set of values of μ on $\mathfrak{D}_\mu$ is the entire unit interval. The proof of the last assertion is based on an intricate number theoretic construction. If α is irrational, $\alpha>1$, $\beta>0$, the set consisting of the integer parts of the numbers $\alpha n+\beta$ is called a quasi-progression. It is proved (by techniques of Diophantine approximations) that every quasi-progression has an infinite number of terms in common with every arithmetic progression, and therefore has inner measure zero and outer measure one.

If D is the ordinary limit density of a set of positive integers, defined on the class $\mathfrak{D}$ of sets for which the limit exists, if ω is the outer measure induced by D and $\mathfrak{D}_\omega$ is the class of sets measurable with respect to ω, then $\mathfrak{D}_\omega=\mathfrak{D}$. The author discusses properties of D and $\mathfrak{D}$ and proves, in particular, the following result: if $\{A_n\}$ is a decreasing sequence of sets in $\mathfrak{D}$ then there exists a set $B\epsilon\mathfrak{D}$ and a sequence $\{F_n\}$ of finite sets such that $B-F_n\subset A_n$ for all n and $D(A_n)\rightarrow D(B)$. It is shown that replacing the $(C, 1)$ convergence occurring in the definition of D by a regular Toeplitz summability method stronger than $(C, 1)$ does not alleviate any of the pathological properties of D.

The paper concludes with a discussion of the customary dyadic mapping $A\rightarrow\Gamma(A)$ sending each set A of positive integers into a point in the unit interval. The author remarks that $\Gamma(\mathfrak{D}_0)$ is the set of rational numbers and raises the question of the measurability (and the value of the measure) of $\Gamma(\mathfrak{D}_\mu)$. *P. R. Halmos* (Chicago, Ill.).

Tornier, Erhard. Mass- und Inhaltstheorien, in denen die Additivität der Masse nur im Unendlichkleinen gefordert wird. Abh. Preuss. Akad. Wiss. Math.-Nat. Kl. **1941**, no. 8, 35 pp. (1941).

The author considers nonnegative, monotone, but not necessarily additive set functions I defined on a field of subsets of a set B, subject to the additional conditions: (i) if $\alpha=\sum_{i=1}^{\infty}\alpha_i$ then $I(\alpha)=\lim_n I(\sum_{i=1}^{n}\alpha_i)$, (ii) there exist two increasing functions f and g of a positive real variable x such that $0<f(x)\leqq g(x)$, $\lim_{x\to 0} g(x)=0$, and, for disjoint sets α, β, $f\{I(\beta)\}\leqq I(\alpha+\beta)-I(\alpha)\leqq g\{I(\beta)\}$. (Additivity corresponds to the case $f(x)=g(x)=x$, identically in x.) A typical example of such a set function is obtained from a countably additive measure m and a strictly increasing continuous function φ by writing $I(\alpha)=\varphi(m(\alpha))$. It is proved that if B contains no atoms then every example is of this type if and only if the conditions $I(\alpha)\geqq I(\alpha')$, $I(\beta)\geqq I(\beta')$, and $\alpha\beta=0$ imply that $I(\alpha+\beta)\geqq I(\alpha'+\beta')$. The main body of the paper reproduces in great detail the theorems involved in the standard construction of Jordan and Lebesgue measures, the principal difference being the systematic use of $I(\sum_{i=1}^{n}\alpha_i)$ in place of $\sum_{i=1}^{n}I(\alpha_i)$.

P. R. Halmos (Chicago, Ill.).

Nakamura, Masahiro, and Sunouchi, Gen-ichirô. Note on Banach spaces. IV. On a decomposition of additive set functions. Proc. Imp. Acad. Tokyo **18**, 333–335 (1942). [MF 14763]

[For note III, by Nakamura, see the same Proc. **18**, 267–268 (1942); these Rev. **7**, 250.] Combining Hausdorff's maximality principle and a proof given by R. S. Phillips [Bull. Amer. Math. Soc. **46**, 274–277 (1940); these Rev. **1**, 240] the authors outline a brief proof of the following general measure theory theorem. Let I be a σ-ideal in a Boolean algebra L and let $x(e)$ be completely additive (c.a.) on L to a Banach space X; then there is a unique decomposition $x(e)=x_1(e)+x_2(e)$, where $x_1(e)\equiv x(a\cap e)$ for some fixed $a\epsilon I$, x_2 vanishes on I, and each x_i is c.a. This generalizes a theorem of Phillips and has been given also by C. E. Rickart [Duke Math. J. **4**, 653–665 (1943); these Rev. **5**, 232]. The theorem also holds (with the present proof) when X is linear normed and c.a. is replaced by additive and s-bounded, as Rickart noted. Corollaries to the present theorem are Phillips' theorem [I is the set of all subsets each having power less than some $\aleph$] and Lebesgue's decomposition theorem [I consists of all sets of measure 0]. The paper concludes with a proof that if $x(e)$ on a Borel field of measurable sets to X is completely additive and vanishes when meas $(e)=0$ then $x(e)$ is absolutely continuous, a theorem due to Dunford [Trans. Amer. Math. Soc. **44**, 305–356 (1938), in particular, p. 333] and later generalized by Rickart [Trans. Amer. Math. Soc. **52**, 498–521 (1942); these Rev. **4**, 162] to the case when X is a locally convex linear topological space.

B. J. Pettis.

Pauc, Christian. Prolongement d'une mesure vectorielle jordanienne en une mesure lebesguienne. C. R. Acad. Sci. Paris **223**, 709–711 (1946).

If V is a weakly complete normed vector space, a finitely additive measure with values in V may be extended to a countably additive one if and only if its original set of values is bounded and it is conditionally countably additive. Two methods of proof are sketched corresponding to the customary Borel (transfinite induction) and Lebesgue (outer measure) procedures.

P. R. Halmos (Chicago, Ill.).

Schärf, Henryk. Intégrale et mesure dans certains espaces algébriques. Supplément. Portugaliae Math. **5**, 142 (1946).

The author observes that an earlier theorem [Portugaliae Math. **4**, 211–216 (1945); these Rev. **7**, 11] identifying the "mean" of a function, when it exists, with a certain linear functional, remains true under weakened hypotheses.

L. H. Loomis (Cambridge, Mass.).

Wendelin, H. Konvergenz- und Häufungsstellensätze nebst Anwendungen auf Darbouxsche Summen. Deutsche Math. **7**, 195–204 (1943).

A somewhat simpler proof of the Darboux theorem for lower integrals is given after proving the following necessary and sufficient condition that a real sequence (a_n) converges to its (proper or improper) least upper bound: corresponding to each positive null sequence (ϵ_n) there is a sequence (R_n) such that, for all r and n, $r\geqq R_n$ implies $a_n\leqq a_r+\epsilon_n$.

J. F. Randolph (Oberlin, Ohio).

Birindelli, C. Sul calcolo dell'integrale di Lebesgue del prodotto di due funzioni e applicazione. I. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **1**, 515–522 (1946).

Divide the n-dimensional Euclidean space into a net of cubes of edge $1/k$ by the hyperplanes $x_i=m/k$, $i=1,\cdots,n$; $m=0,\pm1,\pm2,\cdots$. If g is a Lebesgue integrable function on a measurable set E, define g_k on E to be constant on the part of E in each cube C of the net, having in $C\cdot E$ the average value of g there, $\int_{C\cdot E}g/m(C\cdot E)$. The author gives some sets of sufficient conditions for (1) $\lim_k\int_E g_k f=\lim_k\int_E g f_k=\int_E gf$. For example, it suffices that f and g are of integrable square on E and that g is bounded on E.

M. M. Day.

Birindelli, C. Sul calcolo dell'integrale di Lebesgue del prodotto di due funzioni e applicazioni. II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **1**, 673–680 (1946).

Given $\Phi(x_1,\cdots,x_n)$ periodic in each x_i with period ω_i, Φ quasicontinuous and of integrable square on each bounded measurable subset of the n-dimensional Euclidean space, let f be quasicontinuous on a bounded measurable set E, and suppose, furthermore, that there exist $\alpha,\beta\geqq1$ with $1/\alpha+1/\beta=1$ and $|f|^\alpha$, $|f|^\beta$ summable in E. Then

$$\lim_{m\to\infty}\int_E f(P)\Phi(mP)dP=\left\{\frac{1}{\omega_1\cdots\omega_n}\int_A\Phi(P)dP\right\}\int_E f(P)dP,$$

where A is any period interval $a_i\leqq x_i\leqq a_i+\omega_i$. This and a related result are proved using note I [see the preceding review]. Further sufficient conditions are given for the validity of equation (1) of that note.

M. M. Day.

Tautz, G. Eine Verallgemeinerung der partiellen Integration; uneigentliche mehrdimensionale Stieltjesintegrale. Jber. Deutsch. Math. Verein. **53**, 136–146 (1943).

Let Ω and ω denote bounded B-measurable subsets of a Euclidean space. Let F and G be completely additive set functions vanishing outside Ω, so that $F(\omega\cap\Omega)=F(\omega)$. Let ω_P be the set obtained by translating the set ω parallel to itself so that P becomes the origin of coordinates, $\tilde\omega_P$ the reflection of ω_P on P. Then $F(\omega_P)$ is a B-measurable point function on Ω, and the formula for integration by parts takes the form

$$\int_\Omega F(\omega_P)dG_P=\int_\Omega G(\tilde\omega_P)dF_P,$$

a special case being the well-known formula in one dimension written in the form

$$\int_a^b (F(x)-F(a))dG(x) = \int_a^b (G(b)-G(x))dF(x).$$

This theorem is used to set up a definition of an improper Stieltjes integral $\int_Q f(P)dF_P$. Here $f(P)$ is a point function which can be written $G(\bar{\omega}_P)$, G is totally additive, F is additive on a ring of sets $\mathfrak{R}$ such that, if ω' and ω'' are any two sets of $\mathfrak{R}$, then $F(\omega)$ is uniformly bounded on ω_P' and ω_P'' and $\omega_P' \cap \omega_Q''$; furthermore there exists a sequence of completely additive set functions F_n, depending on ω' and ω'', such that $F_n(\omega_P')$, $F_n(\omega_P'')$, $F_n(\omega_{P_0}' \cap \omega_P'')$, P_0 fixed, are uniformly bounded in n and P and converge to $F(\omega_P')$, $F(\omega_P'')$ and $F(\omega_{P_0}' \cap \omega_{P_0}'')$, respectively.

T. H. Hildebrandt (Ann Arbor, Mich.).

Choquet, Gustave, et Pauc, Christian. Étude des propriétés tangentielles à partir de la notion d'invariance par translation. Bull. Sci. Math. (2) 70, 12–21 (1946).

In E^n let K be a rectifiable curve and M an arbitrary set. The following holds at almost every point p of K: the tangent T of K exists and the point set $C(M, p)$ which carries the contingent of M at p goes into itself under translation parallel to T. This theorem leads to simplifications of proofs given by F. Roger [Acta Math. 69, 99–133 (1938)].

Let $B(M, p)$ consist of the straight lines in $C(M, p)$ through p. The following is proved. The set I of those points p of a closed set M in E^{n+q}, through which an n-dimensional plane disjoint from $B(M, p)-p$ exists, is contained in a countable number of q-dimensional Lipschitz manifolds L_q. At every point p_1 of I, except for a set of q-dimensional Hausdorff measure 0, the set $C(M, p_1)$ goes into itself under translations along q independent lines through p_1 which are tangent to all L_q through p_1.

H. Busemann (Northampton, Mass.).

Viola, T. Sulla definizione della lunghezza d'una curva. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 1, 724–728 (1946).

The author discusses the equality of the ordinary length and the Minkowski length of a simple arc. [See J. Favard, Bull. Soc. Math. France 61, 63–84 (1933).]

H. Federer (Providence, R. I.).

Zwirner, Giuseppe. A proposito di una interpretazione geometrica del lemma fondamentale del calcolo integrale. Rend. Sem. Mat. Univ. Padova 15, 139–143 (1946).

Proof that, if a curve in Euclidean 3-space, a continuous image of a line segment, has a tangent line at each point parallel to the (x, y)-plane, and if the projection of the curve on the (x, y)-plane is rectifiable, then the curve lies in a plane parallel to the (x, y)-plane. The method is to prove that $dx/ds=0$, where s is the arc length of the projection. A second theorem: if $F(x, y)$ is of class C' in an open set D, and $x=\phi(t)$, $y=\psi(t)$ is a curve K in D, with ϕ, ψ continuous and of bounded variation, $t_0 \leqq t \leqq t_1$, and $F_x=F_y=0$ on K, then $F[\phi(t), \psi(t)]$ is constant, $t_0 \leqq t \leqq t_1$. The proof makes use of inscribed polygons.

A. B. Brown (Flushing, N. Y.).

Zwirner, Giuseppe. Sulle radici dei sistemi di equazioni non lineari. Rend. Sem. Mat. Univ. Padova 15, 132–134 (1946).

Short proof of the known theorem that, if $f_i(x_1, \cdots, x_n)$, $i=1, \cdots, n$, are continuous for $|x_i| \leqq 1$, and $f_i(x) \leqq 0$ when $x_i=-1$ and $f_i(x) \geqq 0$ when $x_i=1$, $i=1, \cdots, n$, then there is at least one point with every $|x_i| \leqq 1$ such that $f_i(x)=0$, $i=1, \cdots, n$. The proof makes use of the Kronecker index and a theorem of Hadamard [J. Tannery, Introduction à la Théorie des Fonctions d'une Variable, Paris, 1910, p. 469]. See also Brusotti [Ann. Scuola Norm. Super. Pisa (2) 11, 211–215 (1942); these Rev. 8, 128] and the two following reviews.

A. B. Brown (Flushing, N. Y.).

Scorza-Dragoni, G. Un'osservazione sui sistemi di equazioni algebriche. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 1, 332–335 (1946).

Proof of the same theorem as in the preceding review for the case that the given functions are polynomials, using a theorem of Birkhoff and Kellogg [Trans. Amer. Math. Soc. 23, 96–115 (1922), in particular, pp. 98–99].

A. B. Brown (Flushing, N. Y.).

Scorza Dragoni, Giuseppe. Un'osservazione sulle radici di un sistema di equazioni non lineari. Rend. Sem. Mat. Univ. Padova 15, 135–138 (1946).

Short proof of the same theorem as in the second preceding review, using the Kronecker index. Some additional observations are made.

A. B. Brown (Flushing, N. Y.).

Scorza Dragoni, Giuseppe. Sulla definizione assiomatica dell'area di una superficie. Rend. Sem. Mat. Univ. Padova 15, 8–24 (1946).

The author considers the Lebesgue area $A(S)$ of parametrically given surfaces. As a functional of S, this area $A(S)$ has the following well-known properties. (i) $A(S)$ is nonnegative and lower semi-continuous. (ii) If S_1, S_2 are congruent surfaces, then $A(S_1)=A(S_2)$. (iii) If S is a polyhedron, then $A(S)$ agrees with the area of the polyhedron in the elementary sense. The author defines a certain very restricted class K of surfaces S and proves that any functional $\varphi(S)$ defined for $S \epsilon K$ coincides with $A(S)$ in K provided that $\varphi(S)$ possesses the properties (i), (ii), (iii) and a further rather complicated additivity property.

T. Radó (Columbus, Ohio).

Cesari, Lamberto. Caratterizzazione analitica delle superficie continue di area finita secondo Lebesgue. Ann. Scuola Norm. Super. Pisa (2) 10, 253–295 (1941); 11, 1–42 (1942).

Let Q denote the square $Q: 0 \leqq u \leqq 1$, $0 \leqq v \leqq 1$ and consider a continuous mapping $\Phi: x=x(u, v)$, $y=y(u, v)$, $(u, v) \epsilon Q$. Let C be a Jordan curve in Q. For every point (x, y) in the (x, y)-plane, an index-function $O(x, y, C)$ is defined as follows. If $(x, y) \epsilon \Phi(C)$, then $O(x, y, C)=0$. If $(x, y) \bar{\epsilon} \Phi(C)$, then $O(x, y, C)$ is equal to the topological index of (x, y) with respect to the oriented closed continuous curve that is the image of C under Φ. Now let σ be a generic notation for any finite system $C_1, \cdots, C_m$ of Jordan curves in Q that are mutually exterior to each other and define $\Psi(x, y) = \sup \sum |O(x, y, C_i)|$, $i=1, \cdots, m$, where the least upper bound is taken with respect to all possible systems σ. The function $\Psi(x, y)$ is termed the characteristic function of the mapping Φ. The mapping Φ is of bounded variation if $\Psi(x, y)$ is summable, and the integral of $\Psi(x, y)$ is then defined as the total variation $W(\Phi)$ of Φ. If Φ is not of bounded variation, then one defines $W(\Phi)=\infty$. Now let $S: x=x(u, v)$, $y=y(u, v)$, $z=z(u, v)$, $(u, v) \epsilon Q$, be a representation of a continuous surface S, and denote by Φ_1, Φ_2, Φ_3 the continuous mappings that are obtained by omitting

$z(u, v)$, $y(u, v)$, $x(u, v)$, respectively. Let Ψ_1, Ψ_2, Ψ_3 be the corresponding characteristic functions and let $L(S)$ be the Lebesgue area of S. The principal result of the paper is expressed by the inequality $L(S) \leqq W(\Phi_1) + W(\Phi_2) + W(\Phi_3)$, which yields the theorem that $L(S) < \infty$ if and only if the three projection mappings Φ_1, Φ_2, Φ_3 are of bounded variation.

Let us note that the characteristic function $\Psi(x, y)$ can be shown to agree with the essential multiplicity function introduced by the reviewer, except perhaps on a countable set [for the definition of the essential multiplicity function, see Radó, Duke Math. J. **4**, 189–221 (1938); Radó and Reichelderfer, Trans. Amer. Math. Soc. **49**, 258–307 (1941); these Rev. **2**, 257]. As a consequence, the work of Cesari shows considerable overlapping with similar researches carried on in America during recent years. However, the main result established in the present paper is essentially new.

The proof contains several interesting auxiliary theorems. For example, the representation of S, considered as a mapping, gives rise to a collection G of continua g in Q, each of which is a maximal continuum on which x, y, z are all three constant. Similarly, the projection mappings Φ_1, Φ_2, Φ_3 give rise to collections G_1, G_2, G_3. Let F_1 be the set of those points (x, y) for which the inverse set $\Phi_1^{-1}(x, y)$ has some component g_1 which is not comprised in the collection G and let the sets F_2, F_3 be defined similarly in the (x, z), (y, z) planes, respectively. The author proves the theorem: if Φ_1, Φ_2, Φ_3 are of bounded variation, then the sets F_1, F_2, F_3 are all three of measure zero. [Theorem 4 on page 276 is false. Various other topological details may require more adequate study. However, the reviewer was able to make the necessary adjustments to his own satisfaction.] *T. Radó.*

Cesari, Lamberto. Su di un teorema di T. Radó sulle trasformazioni continue. Ist. Veneto Sci. Lett. Arti. Parte II. Cl. Sci. Mat. Nat. **101**, 377–403 (1942).

The theorem referred to in the title reads as follows. Let $z = f(w)$ be a continuous complex function in a bounded simply connected Jordan region R, with boundary curve C, where $w = u + iv$, $z = x + iy$. Let T denote the continuous mapping defined by the equation $z = f(w)$, $w \in R$. Let z_0 be a point such that the following conditions hold. (i) The topological index of z_0 with respect to the oriented closed continuous curve $T(C)$ is equal to an integer $k \neq 0$. (ii) The set $T^{-1}(z_0)$ consists of a single point w_0 interior to R. Then there exists a $\delta > 0$ such that for $0 < |z - z_0| < \delta$ the point z has at least $|k|$ inverse points in the interior of R. Cesari shows that the assertion remains valid if condition (ii) is dropped and gives a further theorem which gives a more precise insight into the geometrical picture. The methods and results are closely related to those in studies on absolutely continuous plane mappings by T. Radó and P. V. Reichelderfer [Trans. Amer. Math. Soc. **49**, 258–307 (1941); these Rev. **2**, 257]. *T. Radó* (Columbus, Ohio).

Cesari, Lamberto. Sui punti di diramazione delle trasformazioni continue e sull'area delle superficie in forma parametrica. Univ. Roma e Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **3**, 37–62 (1942).

Let R be a simply connected bounded Jordan region in the (u, v)-plane and let $x(u, v)$, $y(u, v)$ be continuous in R. Consider the continuous mapping $\Phi: x = x(u, v), y = y(u, v) \in R$. Let r be a region of the same type contained in R, and let r^* be the positively oriented boundary curve of r. Define $O(x, y, r)$ as follows. If $(x, y) \in \Phi(r^*)$, then $O(x, y, r) = 0$. If $(x, y) \bar{\in} \Phi(r^*)$, then $O(x, y, r)$ is the topological index of (x, y) with respect to the oriented closed continuous curve that is the transform of r^*. Let $o(x, y, r) = 1$ if $O(x, y, r) \neq 0$ and $o(x, y, r) = 0$ if $O(x, y, r) = 0$. Let D be a generic notation for a finite subdivision of R into simply connected Jordan regions, and define $\psi(x, y) = \sup \sum o(x, y, r)$, $r \in D$, and $\Psi(x, y) = \sup \sum |O(x, y, r)|$, $r \in D$, where the least upper bounds are taken with respect to all possible subdivisions D. Clearly $\psi(x, y) \leqq \Psi(x, y)$. A point (x, y) is a branch point (ramification point) for Φ if $\psi(x, y) < \Psi(x, y)$. The author shows that the set of branch points is countable and applies the result to various questions arising in surface area theory. *T. Radó* (Columbus, Ohio).

Cesari, Lamberto. Su di un problema di analysis situs dello spazio ordinario. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) **6**(75), 267–291 (1942).

In Euclidean three-space E_3, let H be a set and C a closed continuous (not necessarily simple) curve. If C can be deformed in H continuously into a single point, then C is said to be homotopic to zero in H. Now let X, Y, Z be three mutually perpendicular lines through a point O and let C be a closed continuous curve in $E_3 - (X + Y + Z)$. Let $\delta > 0$ be the shortest distance between the curve C and the set $X + Y + Z$. Let x, y, z', z'' be four lines satisfying the following conditions. (a) No two have a common point; (b) x is parallel to X, y is parallel to Y, z' and z'' are parallel to Z; (c) the distances between X and x, Y and y, z' and Z, z'' and Z are all less than δ; (d) the lines x, y both intersect the plane strip bounded by z' and z''. The author proves that, if C is homotopic to zero in $E_3 - (x + y + z' + z'')$, then it is also homotopic to zero in $E_3 - (X + Y + Z)$. He states that this problem arose in his work on surface area. *T. Radó.*

Cesari, L. Un complemento alla nota "Criteri di uguale continuità ed applicazioni alla quadratura delle superficie." Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **1**, 292–296 (1946).

[For the paper quoted in the title cf. Ann. Scuola Norm. Super. Pisa (2) **12**, 61–84 (1943); these Rev, **8**, 142.] Let S be a continuous surface, of the type of the circular disc, with finite Lebesgue area $L(S)$. Let K denote the class of all representations $S: x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$, $(u, v) \in C: u^2 + v^2 = 1$, with the following properties. (i) The coordinate functions are absolutely continuous in C in the Tonelli sense. (ii) The first partial derivatives are summable L^2 in C. Assuming that S is nondegenerate, the author states that the generalized conformal maps of S can be obtained and characterized on the basis of the property that in K they minimize the sum of the Dirichlet integrals of the coordinate functions. *T. Radó* (Columbus, Ohio).

Cesari, L. Rappresentazione quasi conforme delle superficie continue. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **1**, 509–514 (1946).

Let a surface S be given by a representation $S: x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$, $(u, v) \in \bar{A}$, where $\bar{A}$ is a bounded simply connected Jordan region. Assume that the Lebesgue area $L(S)$ of S is finite. The author states that he developed a new and direct proof of the theorem of C. B. Morrey on the existence of a generalized conformal map for nondegenerate surfaces [cf. Morrey, Amer. J. Math. **57**, 692–702 (1935); **58**, 313–322 (1936)]. He also states far-reaching generalizations. For example, if $L(S) < \infty$, and if properly defined

asymptotic partial derivatives are used, then there exists a representation upon the unit square Q and a subset H of Q, such that the representation is generalized conformal on H and $L(S)$ is equal to the classical integral taken over H.
T. Radó (Columbus, Ohio).

Youngs, J. W. T. A reduction theorem concerning the representation problem for Fréchet varieties. Proc. Nat. Acad. Sci. U. S. A. **32**, 328–330 (1946).

~~Let f_1 and f_2 be mappings of Peano spaces into the same space and suppose $f_1=l_1m_1$, $f_2=l_2m_2$ are their monotone-light factorizations. A technique recently used by T. Radó [Trans. Amer. Math. Soc. 58, 420–454 (1945), in particular, pp. 425–427; these Rev. 7, 282] is shown to yield the result that f_1 and f_2 are Fréchet equivalent if and only if the middle spaces are homeomorphic and the monotone factors m_1 and m_2 are Fréchet equivalent.~~ *H. Federer.*

* The review is to be replaced by the one in Errata, pp 708

Huskey, Harry D. A note on the area of a nonparametric surface. Bull. Amer. Math. Soc. **52**, 720–726 (1946).

Let $z=f(x, y)$ define on the unit square Q_0 a surface S, absolutely continuous in the sense of Tonelli (ACT) or in that of Young (ACY). Besicovitch [J. London Math. Soc. 19, 138–141 (1944); these Rev. 7, 282] found an ACT surface such that certain successive nearly isosceles right-angled triangulations of Q_0 correspond to inscribed polyhedra whose areas do not tend to that of S. On the other hand, it is shown here, by improving the author's theorems on the problem of Geöcze [Duke Math. J. 11, 333–339 (1944); these Rev. 6, 45] that, in the case of an ACY surface, if certain successive exactly isosceles right-angled triangulations of the plane are translated through the same vector (u, v), then the corresponding triangulations of Q_0, suitably completed near the edges of Q_0, give rise for almost every (u, v) to inscribed polyhedra whose areas tend to that of S. In the special case of an ACT surface, the successive triangulations of the plane can be chosen independently of the surface S.
L. C. Young (Cape Town).

Theory of Series

Everett, C. J. Representations for real numbers. Bull. Amer. Math. Soc. **52**, 861–869 (1946).

The author extends the algorithm by which real numbers are represented as decimals of base $p\geqq 2$, by taking a continuous strictly increasing function $f(t)$ on $0\leqq t\leqq p$ for which $f(0)=0$, $f(p)=1$. If γ_0 denotes any real nonnegative number, we construct a sequence of integers $c_0, c_1, \cdots$ $(0\leqq c_n\leqq p-1$ when $n\geqq 1)$ by the following process:

$$\gamma_0=c_0+f(\gamma_1),\quad c_0\leqq\gamma_0<c_0+1,\quad 0\leqq\gamma_1<p,$$
$$\gamma_1=c_1+f(\gamma_2),\quad c_1\leqq\gamma_1<c_1+1,\quad 0\leqq\gamma_2<p,$$

and so on. The cases when the sequence terminates in $0, 0, \cdots$ or in $p-1, p-1, \cdots$ are studied. Weak sufficient conditions are given for the correspondence between the numbers γ_0 and the sequences $\{c_n\}$ to be one-one. The many-one case is studied; in this case the algorithm defines a set of limit numbers which is perfect and nowhere dense and which is closely related to the well-known Cantor set. Finally the author studies the relation between these investigations and topological transformations of the unit interval into itself.
J. F. Koksma (Amsterdam).

Egerváry, E. A remark on the length of the circle and on the exponential function. Acta Univ. Szeged. Sect. Sci. Math. **11**, 114–118 (1946).

Elementary proof of the theorem that the length of the inscribed polygon in a circle tends to a limit when the number of sides increases indefinitely in such a way that all the sides tend to zero. Parallel proof of the analogous theorem that, if $\alpha_1, \cdots, \alpha_n$ denote positive numbers whose sum is 1, the product $\prod_{\nu=1}^{n}(1+\alpha_\nu)$ tends to a limit when n increases in such a way that all the numbers α_ν tend to zero.
J. F. Koksma (Amsterdam).

Aubert, Karl E. Summation of some series of binomial coefficients on the basis of Cauchy's integral formula. Norsk Mat. Tidsskr. **27**, 76–86 (1945). (Norwegian)

Tenca, Luigi. Espressioni del termine generale di una progressione aritmetica d'ordine m in funzione dei primi $m+1$ termini. Boll. Mat. (4) **3**, 67–70 (1942).

Campagne, C. Euler's summation formula. Verzekerings-Arch. **23**, 81–94 (1942). (Dutch)

Frazer, H. Note on Hilbert's inequality. J. London Math. Soc. **21**, 7–9 (1946).

Hilbert's inequality is

$$\sum_{s=0}^{m}\sum_{t=0}^{m}\frac{a_sa_t}{s+t+1}\leqq\pi\sum_{p=0}^{m}a_p^2.$$

The author improves the constant π to $(m+1)\sin\pi/(m+1)$.
R. P. Boas, Jr. (Providence, R. I.).

Tagamlitzki, Yaroslav. Sur les suites vérifiant certaines inégalités. C. R. Acad. Sci. Paris **223**, 940–942 (1946).

If the complex sequence $\{a_n\}_0^\infty$ satisfies $|\Delta^n a_\nu|\leqq|\Delta^n q^\nu|$ $=q^\nu(1-q)^n$, $0<q<1$; $\nu=0, 1, \cdots$; $n=0, 1, \cdots$, then $a_\nu=Aq^\nu$, $|A|\leqq 1$. Corollary: if $|f^{(n)}(x)|\leqq e^{-x}$, $n=0, 1, \cdots$; $0\leqq x<\infty$, then $f(x)=Ae^{-x}$, $|A|\leqq 1$.
R. P. Boas, Jr.

Pettineo, B. Estensione di una classe di serie divergenti. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **1**, 680–685 (1946).

Let $u_1+u_2+\cdots$ be a divergent series of positive terms, with partial sums $s_n=u_1+\cdots+u_n$. It is well known that, when the terms u_n are bounded, the series

$$\sum\frac{u_n}{s_n},\quad\sum\frac{u_n}{s_n\log s_n},\quad\sum\frac{u_n}{s_n\log s_n\log\log s_n},\cdots$$

are all divergent. This is extended in the following way. Let $P(x)$ denote the product of x and all of the numbers $\log x$, $\log\log x$, $\log\log\log x$, $\cdots$ which are greater than 1. If $\limsup u_n/s_n<1$, then the series $\sum u_n/P(s_n)$ is divergent.
R. P. Agnew (Ithaca, N. Y.).

Bosanquet, L. S. Note on Hölder means. J. London Math. Soc. **21**, 11–15 (1946).

Two theorems are given which answer questions on total inclusion among Hölder, Cesàro and Abel methods of summability. If $s\geqq 2$, there is a series for which the Hölder transform of order 2 diverges to $+\infty$ while the Cesàro transform of order s fails to do so. There is a series for which the Hölder transform of order 2 diverges to $+\infty$ while the Abel transform fails to do so.
R. P. Agnew.

Piranian, George. A summation matrix with a governor. Bull. Amer. Math. Soc. 52, 882–889 (1946).

Corresponding to the formal series $\sum_0^\infty a_n$ let $s_r(a)=\sum_0^r a_k$, $p_r(a)=\sum_0^r|a_k|$, and $P_n(a)=\sum_0^n p_r$. If the Nörlund matrix $(p_{n-r}(a)/P_n(a))\equiv(A_{nr}(a))$ is regular (that is, if $A_{nn}(a)\to 0$) and if the sequence $\{s_r(a)\}$ is summable to S by means of this matrix, then the series $\sum a_n$ is said to be summable (G) to S. The novelty of the method (G) consists in the fact that the matrix $(A_{nr}(a))$ is constructed in each case from the terms of the series in question. The author first mentions four properties of (G) that follow at once from known facts concerning Nörlund and Abel summability. He then proceeds to his main results, among which we mention the following. (i) In order that $A_{nn}(a)\to 0$ it is necessary that $\liminf |a_n|/p_n=0$, and sufficient (but not necessary) that $\lim a_n/p_n=0$. (ii) The method (G) includes the method of convergence. (iii) If $A_{nn}(a)\to 0$, the insertion of a finite number of zero terms in $\sum a_n$ can neither destroy nor create summability (G) for the series, nor alter the value of its (G)-sum. (iv) If $f(z)$ is a polynomial in z with real coefficients then the series $\sum_0^\infty(-1)^n f(n)$ is summable (G).

J. D. Hill (East Lansing, Mich.).

Birindelli, C. Su una generalizzazione della convergenza in media. I. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 1, 325–332 (1946).
Birindelli, C. Su una generalizzazione della convergenza in media. II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 1, 526–530 (1946).

Theorems on convergence in the mean are extended to general summability in the sense of M. Picone [Ann. Mat. Pura Appl. (4) 2, 263–295 (1925)]. The second paper gives applications to the special case of Cesàro summability.

O. Szász (Cincinnati, Ohio).

Sargent, W. L. C. A mean value theorem involving Cesàro means. Proc. London Math. Soc. (2) 49, 227–240 (1946).

Let $0<\lambda\leqq 1$, $a<b<\beta$, $f\in L(a,b)$. The inequality

$$\left|\int_a^b(\beta-t)^{\lambda-1}f(t)dt\right|\leqq\sup_{a<x<b}\left|\int_a^x(x-t)^{\lambda-1}f(t)dt\right|,$$

due to M. Riesz, is very useful in the theory of typical means [see Hardy and Riesz, The General Theory of Dirichlet's Series, Cambridge University Press, 1915]. The author obtains a corresponding result involving C_λ-means,

$$C_\lambda(f,a,b)=\int_a^b(b-t)^{\lambda-1}f(t)dt\Big/\int_a^b(b-t)^{\lambda-1}dt.$$

It is shown that, with the previous assumptions concerning λ, a, b, β, the ratio

$$(*)\qquad\int_a^b(\beta-t)^{\lambda-1}f(t)dt\Big/\int_a^b(\beta-t)^{\lambda-1}dt$$

is contained between $\inf_{a<x<b}C_\lambda(f,a,x)$ and $\sup C_\lambda(f,a,x)$ (inf and sup always mean essential bounds). If $\lambda>1$, $a<b<\beta$, then the absolute value of $(*)$ does not exceed a fixed multiple, depending on λ only, of the larger of the two numbers $\sup_{a<x<b}|C_\lambda(f,a,x)|$, $\sup_{a<x<b}|C_\lambda(f,b,x)|$.

A. Zygmund (Philadelphia, Pa.).

Rankin, R. A. A note on a particular type of asymptotic series. Philos. Mag. (7) 36, 860–861 (1945).

In this note the author shows how the useful range of application of a general class of asymptotic formulae can be extended by obtaining narrower bounds for the remainder. An alternating asymptotic series of the form

$$f(x)=\sum_{\nu=0}^{n-1}(-1)^\nu u_\nu(x)+(-1)^n R_n(x),$$

where $u_\nu(x)$ and $R_n(x)$ are positive functions of x for $x>a$, is called "strictly" asymptotic if $0\leqq R_n(x)\leqq u_n(x)$. If $f(x)$ is expressible in the form

$$f(x)=\int_0^\infty(1+t)^{-1}\varphi(t,x)dt$$

with $\varphi(t,x)\geqq 0$ for $x>a$ and if

$$u_\nu(x)=\int_0^\infty t^\nu\varphi(t,x)dt$$

exists as a convergent integral for all $\nu\geqq 0$, it is shown that $\max[0;u_{n-\frac12}(x)-\frac14\{u_n(x)+u_{n-1}(x)\}]\leqq R_n(x)\leqq\frac12 u_{n-\frac12}(x)$ and, more generally, $0\leqq R_n(x)\leqq\alpha^\alpha(1-\alpha)^{1-\alpha}u_{n-\alpha}(x)$, $0<\alpha<1$.

S. C. van Veen (Delft).

Cesari, L. Sulla moltiplicazione delle serie doppie. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 1, 289–292 (1946).

Statements, without proofs, of three theorems on Cesàro and restricted Cesàro summability of Cauchy products of double series.

R. P. Agnew (Ithaca, N. Y.).

Sheffer, I. M. Note on multiply-infinite series. Bull. Amer. Math. Soc. 52, 1036–1041 (1946).

Let $\sum b(n_1,\cdots,n_k)$ be a multiple series which converges to B. In order that the Cauchy-product series $\sum c(n_1,\cdots,n_k)$ of $\sum b(n_1,\cdots,n_k)$ and $\sum a(n_1,n_2,\cdots)$ converge to AB whenever $\sum a(n_1,\cdots,n_k)$ converges absolutely and converges to A, it is necessary and sufficient that the series $\sum b(n_1,\cdots,n_k)$ have bounded partial sums.

R. P. Agnew.

Fourier Series and Generalizations, Integral Transforms

***Denjoy, Arnaud. Leçons sur le Calcul des Coefficients d'une Série Trigonométrique. Tome I. La Différentiation Seconde Mixte et Son Application aux Séries Trigonométriques.** Gauthier-Villars, Paris, 1941. xiv+84 pp.
***Denjoy, Arnaud. Leçons sur le Calcul des Coefficients d'une Série Trigonométrique. Tome II. Métrique et Topologie d'Ensembles Parfaits et de Fonctions.** Gauthier-Villars, Paris, 1941. 143 pp. [paged 85–227].
***Denjoy, Arnaud. Leçons sur le Calcul des Coefficients d'une Série Trigonométrique. Tome III. Détermination d'une Fonction Continue par ses Nombres Dérivés Seconds Généralisés Extrêmes Finis.** Gauthier-Villars, Paris, 1941. 98 pp. [paged 229–326].

The classical result of Cantor asserts that, if a function $f(x)$ is represented by an everywhere convergent trigonometric series

$$(*)\qquad \tfrac12 a_0+\sum_{n=1}^\infty(a_n\cos nx+b_n\sin nx),$$

this series is unique. If the function $f(x)$ is Lebesgue integrable then, as shown by de la Vallée Poussin [Acad. Roy. Belgique. Bull. Cl. Sci. (5) 2, 702–718 (1912)], the series $(*)$ is the Fourier series of $f(x)$, that is, the coefficients a_n, b_n are expressible by the classical Fourier formulas. In

five notes [C. R. Acad. Sci. Paris 172, 653–655, 833–835, 903–906, 1218–1221; 173, 127–129 (1921)] Denjoy stated that with a suitable definition of integral ("trigonometric integral") an everywhere convergent trigonometric series is always the Fourier series of its sum. This is an analogue of another and earlier result of Denjoy to the effect that with a suitable definition of integral (usually called the Denjoy-Perron integral) an everywhere differentiable function is the integral of its derivative. The problem of the trigonometric integral is, however, much more difficult. The difficulties arise from the fact that, even if (*) converges everywhere, the termwise integrated series, that is,

$$\tfrac{1}{2}a_0x+\sum(a_n \sin nx-b_n \cos nx)/n,$$

is only convergent almost everywhere, and its sum $F(x)$ (even if it exists everywhere) need not have the sum $f(x)$ of (*) for derivative. If, however, we integrate (*) formally twice, the resulting series

$$\tfrac{1}{4}a_0x^2-\sum(a_n \cos nx+b_n \sin nx)/n^2$$

converges uniformly to a continuous function $\Phi(x)$ having $f(x)$ for its second symmetric derivative [Riemann]. In other words,

$$f(x)=\lim_{h\to 0}\{\Phi(x+h)+\Phi(x-h)-2\Phi(x)\}/h^2$$

for all x. Thus it is natural to skip the first integral of the function f and try to define only the second integral. Such an integral ("the second primitive") would have to have the property that every finite symmetric derivative of a continuous function is integrable. The integral would be defined up to an additive linear function which would play here the same role as the constant of integration in ordinary integrals. The notes of Denjoy dealing with the topic were written concisely and many statements were given without proof, making it difficult for the reader to reconstruct the whole argument. In 1938 the author lectured on the topic at Harvard; the present book is an elaboration of these lectures. [It may be added that the initial notes of Denjoy created considerable interest and influenced a number of papers devoted to the theory of the trigonometric integral. See, for example, S. Verblunsky, Fund. Math. 23, 193–236 (1934); Burkill, J. London Math. Soc. 11, 43–48 (1936); J. Marcinkiewicz and A. Zygmund, Fund. Math. 26, 1–43 (1936)].

The book is to consist of five parts of which so far only three have appeared. Not all the book, however, is devoted to the theory of trigonometric integration. Much of the space is given to other topics from the theory of real variables, partly because they are useful in the main problem and partly because of the personal interest of the author. Volume one covers rather familiar ground and consists of two chapters. One of these is devoted to the proof of de la Vallée Poussin's result stated above, treated, however, from the point of view of the second primitive. In the other, differential properties of the function $\Phi(x)$ are investigated and, in particular, various refinements of the theorem of Riemann [see above] are given.

Of the two parts constituting volume two, the first is devoted to a study of geometric properties of linear perfect sets based on the notions of the "index" of a point, and of the "coefficient of isolation" of a portion, of such a set. The index of a point θ of a perfect set P is the limit superior of the ratios $|b_n-\theta|/|a_n-\theta|$ when the interval (a_n, b_n) contiguous to P approaches θ (from the left or from right; in each case we assume that $|a_n-\theta|<|b_n-\theta|$). Let ρ be an isolated portion of P, which means that ρ is contained between two intervals u and v contiguous to P. If ρ, u, v also mean the span of ρ, and the lengths of u, v, then the smaller of the two numbers u/ρ, v/ρ is called the coefficient of isolation of the portion ρ. Between the indices of points and the coefficients of isolation there exist simple relations which are important for the theory of the second symmetric derivative. They are investigated in detail by the author. The major part of the remaining chapter of volume two is devoted to various aspects of the well-known theorem of Baire concerning the structure of the sets of second category and of the functions which are limits of continuous functions.

Volume three consists almost entirely of new results. The first part gives differential properties of functions $F(x)$ such that $|F(x+u)+F(x-u)-2F(x)|\leqq\psi(u)$, where $\psi(u)$ tends to 0 with u. One of the main results here is that, if for every x belonging to a set E of positive measure we have $F(x+h)+F(x-h)-2F(x)=O(h^2)$ as $h\to 0$, then the second symmetric derivative of F exists almost everywhere in E. This symmetric derivative is almost everywhere in E equal to the asymptotic derivative of the ordinary derivative $F'(x)$. The last part of volume three is devoted to the constructive theory of the second primitive. As in his earlier theory of totalization, the author defines a number of standard operations which, starting from Lebesgue integration, define the primitive over wider and wider sets, until it becomes known everywhere. A more complete presentation of the theory is promised for subsequent volumes of the book. *A. Zygmund* (Philadelphia, Pa.).

Sinvhal, S. D. On the points of convergence of Singh's example. Proc. Benares Math. Soc. (N.S.) 7, 35–50 (1945).

This is a study of the points of convergence of the Fourier series of a certain continuous function first studied by A. N. Singh [Rend. Circ. Mat. Palermo 59, 261–264 (1935)]. Singh proved that the series diverges on a certain unenumerable everywhere dense set. The present author defines everywhere dense sets related to that of Singh and proves that the series converges on these sets. *H. E. Bray* (Houston, Tex.).

Alexits, Georges. Sur l'ordre de grandeur de l'approximation d'une fonction par les moyennes de sa série de Fourier. Mat. Fiz. Lapok 48, 410–422 (1941). (Hungarian. French summary)

The main result of this paper is the following theorem. Suppose that a function $f(x)$ of period 2π satisfies uniformly an ordinary Lipschitz condition; denote by $\tilde{f}$ the conjugate of f, and by $\tilde{\sigma}_n^\delta$ the nth Cesàro mean of order $\delta>0$ of the conjugate Fourier series; then (*) $\tilde{\rho}_n^\delta=O(n^{-1})$ as $n\to\infty$, where $\tilde{\rho}_n^\delta=\max|\tilde{f}(x)-\tilde{\sigma}_n^\delta(x)|$ for $0\leqq x\leqq 2\pi$. For $\delta=1$ this was proved (in a different way) independently by A. Zygmund [Bull. Amer. Math. Soc. 51, 274–278 (1945); these Rev. 6, 265]. Alexits also proves the converse: if (*) holds for each $\delta\geqq 1$, then $f(x)$ satisfies a Lipschitz condition of degree 1. *O. Szász* (Cincinnati, Ohio).

v. Sz. Nagy, Béla. Approximation der Funktionen durch die arithmetischen Mittel ihrer Fourierschen Reihen. Mat. Fiz. Lapok 49, 123–138 (1942). (Hungarian. German summary)

The results of this paper are in essence contained in another paper by the same author [Acta Univ. Szeged. Sect. Sci. Math. 11, 71–84 (1946); these Rev. 8, 150]. *O. Szász* (Cincinnati, Ohio).

The results announced in the paper are proved in a paper reviewed earlier. [Trans. Amer. Math. Soc. 56, 508–518 (1944); these Rev. 6, 126]

Loo, Ching-Tsün. On the Cesaro means of Fourier series. Acad. Sinica Science Record 1, 341–348 (1945).

Let $f(x)\epsilon L$ have period 2π and denote by σ_n^α the Cesàro sum of order α of the Fourier series of $f(x)$, where $\alpha\geqq 0$. Let $\varphi(t)=\frac{1}{2}\{f(x+t)+f(x-t)-2s\}$,

$$\varphi_p(t)=pt^{-p}\int_0^t (t-u)^{p-1}\varphi(u)du, \qquad p>0.$$

By a theorem of Paley, if $\sigma_n^\alpha\to s$, then, as $t\to 0$, $\varphi_{\alpha+1+\delta}(t)\to 0$ for every $\delta>0$; the result is no longer true if we replace δ by zero. The main result of the paper is that, if $\alpha>0$ and $\sigma_n^\alpha-s=o(1/\log n)$, then $\varphi_{\alpha+1}(t)\to 0$ as $t\to 0$. This result thus generalizes a theorem of Hardy and Littlewood which deals with the case $\alpha=0$. *R. Salem* (Cambridge, Mass.).

Obreschkoff, N. Über die C-Summierbarkeit der derivierten Reihen der Fourierschen Reihen. Abh. Preuss. Akad. Wiss. Math.-Nat. Kl. 1941, no. 15, 28 pp. (1942).

Let $f(t)$ be Lebesgue integrable in $(-\pi, \pi)$, with Fourier series $\sum(a_n \cos nt+b_n \sin nt)$. The series obtained by differentiating the Fourier series formally p times at $t=x$ is (*) $\sum n^p\{a_n \cos(nx+\frac{1}{2}p\pi)+b_n \sin(nx+\frac{1}{2}p\pi)\}$. Let $P(t)$ denote a polynomial of degree p, for which $P^{(p)}(t)=s$, and write $\mu(t)=f(x+t)-P(t)+(-1)^p\{f(x-t)-P(-t)\}$. Let $\mu_0(t)=\mu(t)$ and

$$\mu_\alpha(t)=t^{-\alpha}\int_0^t (t-u)^{\alpha-1}\mu(u)du, \qquad \alpha>0.$$

The author proves the following theorems. (1) If $\alpha\geqq 0$ and, for a suitable $P(t)$,

$$\mu_\alpha^*(t)=t^{-1}\int_0^t |\mu_\alpha(u)|du=o(t^p), \qquad t\to+0$$

(or $\mu_\alpha^*(t)=O(t^p)$ and $\mu_{\alpha+1}(t)=o(t^p)$), then the series (*) is summable (C, k) to s for $k>\alpha+p$. (2) If $\alpha\geqq 0$, $0<\omega\leqq 1$ and $\mu_\alpha^*(t)=O(t^{p+\omega})$, then $c_n^k=O(n^{-\omega})$ for $p+\alpha+1\geqq k>p+\alpha+\omega$, $c_n^k=O(n^{-\omega}\log n)$ for $k=p+\alpha+1$, where c_n^k is the nth partial Cesàro mean of order k for the series (*). [Here the condition $p+\alpha+1\geqq k$ may be omitted.] (3) If $k\geqq p$ and (*) is summable (C, k) to s, then for a suitable $P(t)$, $\mu_\alpha(t)=o(t^p)$ for $\alpha>k-p+1$. (4) If $\alpha\geqq 0$ and $t^{-p}\mu_\alpha(t)$ is of bounded variation in $(0, \pi)$, then (*) is summable $|C, k|$ to s for $k>\alpha+p$. (5) If $k\geqq p$ and (*) is summable $|C, k|$, then $t^{-p}\mu_\alpha(t)$ is of bounded variation in $(0, \pi)$ for $\alpha>k-p+1$.

Thus, in particular, we have the following corollaries. (i) A necessary and sufficient condition for (*) to be summable (C) to s is that $t^{-p}\mu_\alpha(t)=o(1)$ for some α. (ii) A necessary and sufficient condition for (*) to be summable $|C|$ is that $t^{-p}\mu_\alpha(t)$ is of bounded variation in $(0, \pi)$ for some α.

The author proved theorem 1 in the case $p=1$ in an earlier paper [Bull. Soc. Math. France 62, 84–109, 167–184 (1934)]. Theorem 3 was also obtained by the reviewer [Proc. London Math. Soc. (2) 46, 270–289 (1940), lemma 9; these Rev. 1, 329] in the course of his proof that a necessary and sufficient condition for (*) to be summable (C, k), where $k\geqq p$, is that, for a suitable polynomial $P(t)$, the function $g(t)=t^{-p}\mu(t)$ is integrable in the Cesàro-Lebesgue sense and its Fourier series is summable $(C, k-p)$ at $t=0$.

Some cases at least of theorem 1 follow from this criterion by means of known results. Thus, for example, suppose that α is an integer and $\mu_\alpha(t)=o(t^p)$. Then (a) it follows that $g(t)$ is integrable (CL) and $g(t)=o(1)(C, \alpha)$ [Bosanquet, Quart. J. Math., Oxford Ser. 10, 67–74 (1939), lemma 1]; (b) it then follows that the Fourier series of $g(t)$ is summable $(C, \alpha+\delta)$ at $t=0$ [Burkill, J. London Math. Soc. 10, 254–259 (1935); Bosanquet, loc. cit. (1940), lemma 10]; (c) hence it follows from the reviewer's criterion that (*) is summable $(C, \alpha+p+\delta)$. In particular, these results include corollary (i).

Theorem 5 was also obtained by the reviewer in the case $p=1$, k a positive integer, $\alpha=k+1$ [Quart. J. Math., Oxford Ser. 12, 15–25 (1941), lemma 4; these Rev. 2, 361] in establishing a criterion for the $|C|$ summability of the first derived Fourier series, and the case $p=1$, α a positive integer, $k=\alpha+2$ of theorem 4 may be deduced from this criterion [cf. Bosanquet, loc. cit. (1941), lemmas 5 and 6]. These results include corollary (ii). *L. S. Bosanquet* (London).

Geronimus, J. On positive trigonometric polynomials and harmonic functions. C. R. (Doklady) Acad. Sci. URSS (N.S.) 51, 569–572 (1946).

A nonnegative trigonometric polynomial $H_n(\varphi)$ of order n admits the representation $H_n(\varphi)=|\pi_n(e^{i\varphi})|^2$, $\pi_n(z)$ being a polynomial of degree n in z, nonvanishing for $|z|<1$, with $\pi_n(0)>0$. Assuming $H_n(\varphi)$ to be not only nonnegative, but even positive for all φ, the author obtains new methods of finding $\pi_n(z)$ and gives some new estimates for $H_n(\varphi)$. His results, stated without proof, depend on the use of the determinants $\Delta_k=|C_{i-j}|_0^k$, $k=0, 1, \cdots, n$, where

$$C_k=(2\pi)^{-1}\int_0^{2\pi}\{H_n(\varphi)\}^{-1}e^{-ik\varphi}d\varphi.$$

It is also stated that if $\sum(\Delta_k^2-\Delta_{k+1}\Delta_{k-1})^{\frac{1}{2}}/\Delta_k<\infty$ then $G(\varphi)=\frac{1}{2}C_0+\Re\sum_{k=1}^\infty C_ke^{ik\varphi}$ is the Fourier series of a continuous positive function. Best possible bounds for $G(\varphi)$ are given in terms of the Δ_k. *H. Pollard* (Ithaca, N. Y.).

Ilieff, Lübomir. Über trigonometrische Polynome mit monotoner Koeffizientenfolge. Jber. Deutsch. Math. Verein. 53, 13–23 (1943).

The principal results of this paper go slightly beyond the first part of theorem 6.5.1 of Szegö, Orthogonal Polynomials [Amer. Math. Soc. Colloquium Publ., v. 23, New York, 1939; these Rev. 1, 14], p. 131. *G. Szegö.*

Nikolsky, S. On interpolation and best approximation of differentiable periodic functions by trigonometrical polynomials. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 10, 393–410 (1946). (Russian. English summary)

Let $W^{(r)}$ be the class of functions of period 2π which have an absolutely continuous $(r-1)$th derivative and for which $|f^{(r)}(x)|\leqq 1$ almost everywhere. It has been proved by Favard [Bull. Sci. Math. (2) 61, 209–224, 243–256 (1937)] and by Achyeser and Krein [C. R. (Doklady) Acad. Sci. URSS (N.S.) 15, 107–111 (1937)] that, if $E_n[f]$ is the best approximation of f by trigonometric polynomials of order n, then sup $E_n[f]$ for $f\epsilon W^{(r)}$ is equal to A_rn^{-r}, where A_r is a positive constant depending on r only. In particular, (*) $\limsup_{n\to\infty} n^rE_n[f]\leqq A_r$ for every $f\epsilon W^{(r)}$. The author shows that there is an $f\epsilon W^{(r)}$ which gives equality in (*). He also shows that, if $\bar{S}_n(x, f)$ is the trigonometric polynomial of order n coinciding with f at the $2n+1$ points $2k\pi/(2n+1)$, $k=0, 1, \cdots, 2n$, then

$$\limsup_{n\to\infty} (n^r/\log n)|f(x)-\bar{S}_n(x, f)|\leqq(2/\pi)A_r\lambda(x),$$

where $\lambda(x)=\limsup_{n\to\infty} \sin|(n+\frac{1}{2})x|$. For every x there is an $f\epsilon W^{(r)}$ giving equality here. *A. Zygmund.*

Gourevitch, V. Sur certains cas de coïncidence du polynôme-minimum trigonométrique et des polynômes d'approximation quadratique et d'autres degrés. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **10**, 469–486 (1946). (Russian. French summary)

Given any function $f(x)$ of period 2π and any $2n+1$ distinct (mod 2π) points, there is always a unique trigonometric polynomial of order n coinciding with f at these points. If we have $m>2n+1$ points $0\leqq x_1<x_2<\cdots<x_m<2\pi$ and if $p\geqq 1$ is fixed, there is a trigonometric polynomial $S(x)$ of order n making

$$(*)\qquad \left\{\sum_{k=1}^{m}|f(x_k)-S(x_k)|^p\right\}^{1/p}$$

a minimum. For $p>1$ this polynomial is unique. The case $p=\infty$ is of particular interest; it means that $S(x)$ minimizes $\max_k|f(x_k)-S(x_k)|$. The case $m=2n+2$ is investigated in the present paper. Among others the following results are proved. (1) If the points $x_1,\cdots,x_{2n+2}$ are equally spaced, then the minimizing polynomials $S(x)$ are the same for every p, $1\leqq p\leqq\infty$. More generally, (2) for any $2n+2$ points $0\leqq x_1<\cdots<x_{2n+2}<2\pi$, if there is no polynomial of order n coinciding with f at these points, a necessary and sufficient condition that the same polynomial S should simultaneously minimize $(*)$ for $p=\infty$ and for some finite value of $p\geqq 1$ is that

$$x_3-x_1=x_5-x_3=\cdots=x_4-x_2=x_6-x_4=\cdots=\pi/(n+1).$$

(3) If for any $m=2n+2$ distinct points x_k the same polynomial S of order n minimizes $(*)$ for two distinct values of $p\geqq 1$, then it also minimizes $(*)$ for any other value of $p\geqq 1$.
A. Zygmund (Philadelphia, Pa.).

Chandrasekharan, K. On multiple Fourier series. Proc. Indian Acad. Sci., Sect. A. **24**, 229–232 (1946).

The author announces many new results on summability of multiple Fourier series by spherical means. In analogy to the means $S_R^\delta(x)=\sum_{\nu\leq R}(1-\nu^2/R^2)^\delta e^{i(n_1x_1+\cdots+n_kx_k)}$ for any δ, the author introduces for the given function $f(x)$ the spherical means

$$f_p(x,t)=ct^{-k}\int f(x+y)(1-s^2/t^2)^{p-1}dy,\qquad p>0,$$

where dy is the k-dimensional volume element,

$$\sum_1^k(y_i-x_i)^2=s^2\leqq t^2,$$

and c is a constant. In the limiting case $p=0$, $f_0(x,t)$ is the reviewer's original mean $c\int f(x+t\xi)dw_\xi$, and can be defined only thus. While the reviewer correlated the behavior of S_R^δ only with $f_0(x,t)$, the author takes in $f_p(x,t)$ for any p and obtains many theorems. In particular, he gives the following extension to all k of a theorem of Hardy and Littlewood. At a point x, $\lim_{R\to\infty}S_R^\delta$ exists for some δ if and only if $\lim_{t\to 0}f_p(x,t)$ exists for some p. *S. Bochner.*

Fejes, L. Über die Fouriersche Reihe der Abkühlung. Acta Univ. Szeged. Sect. Sci. Math. **11**, 28–36 (1946).

Let $0=\mu_0,\mu_1,\cdots$ be the nonnegative roots of the equation $z+h\tan\pi z=0$, $h>0$, arranged in increasing order. Suppose $f(x)\epsilon L(a,b)$, where $b-a=2\pi$. Then the nonorthogonal series

$$f(x)\sim\sum_{\nu=0}^{\infty}(a_\nu\cos\mu_\nu x+b_\nu\sin\mu_\nu x),$$

where

$$a_0=\frac{h}{2(1+\pi h)}\int_a^b f(t)dt,$$

$$\begin{matrix}a_\nu\\ b_\nu\end{matrix}=\frac{2\mu_\nu}{2\pi\mu_\nu-\sin 2\pi\mu_\nu}\int_a^b f(t)\begin{matrix}\cos\mu_\nu t\\ \sin\mu_\nu t\end{matrix}\,dt$$

is called the "Fouriersche Reihe der Abkühlung" of $f(x)$.

It is proved that if a series $\sum(a_\nu\cos\mu_\nu x+b_\nu\sin\mu_\nu x)$ has partial sums with the properties (a) $\lim_{n\to\infty}\{s_n(a)+s_n(b)\}=0$, (b) $\int_a^x s_n(t)dt$ converges uniformly to the indefinite integral of a function $f(x)$ of $L(a,b)$, then it is the Fourier series of $f(x)$ in the preceding sense. Without these conditions such a series may converge everywhere on (a,b) to zero without vanishing identically. *H. Pollard* (Ithaca, N. Y.).

Gagaeff, B. Sur quelques classes de fonctions orthogonales. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **10**, 197–206 (1946). (Russian. French summary)

The following theorem is proved. Suppose the functions $\varphi_n(x)$ are orthogonal in $a\leqq x\leqq b$ with respect to a function $q(x)$ and that their derivatives are orthogonal with respect to a function $p(x)$, so that $\int_a^b q(x)\varphi_n(x)\varphi_m(x)dx=0$, $n\neq m$, and $\int_a^b p(x)\varphi_n'(x)\varphi_m'(x)dx=0$, $n\neq m$. Suppose, furthermore, that the functions $(p(x)\varphi_n'(x))'$, $q(x)$, $\varphi_n(x)$ are continuous, that the system $\{\varphi_n'(x)\}$ is closed and that the system $\{\varphi_n\}$ contains unity. Suppose also that $p(x)$ and $q(x)$ vanish only at $x=a,b$ and that $(p(x))^{-1}$ is integrable over $a\leqq x\leqq b$. Then the system $\{\varphi_n(x)\}$ is unique and satisfies the following relations: $p(a)\varphi_n'(a)=p(b)\varphi_n'(b)=0$ and $(p(x)\varphi_n'(x))'+\lambda_nq(x)\varphi_n(x)=0$, where

$$\lambda_n^{-1}=\int_a^b q(x)(\varphi_n(x))^2dx.$$

The condition that $(p(x))^{-1}$ is integrable for $a\leqq x\leqq b$ is then relaxed.

If $p(x)\equiv q(x)\equiv 1$, $a=0$ and $b=2\pi$, then $\varphi_n(x)=\cos\frac{1}{2}nx$, $n=0,1,2,\cdots$. If $p(x)\equiv q(x)\equiv x$, $a=0$, $b=1$, then $\varphi_n(x)=I_0(\lambda_nx)$, where $I_0(x)$ is Bessel's function and λ_n are the roots of $I_1(\lambda)=0$. *A. C. Offord.*

Ríos, S. Notes of priority. Revista Mat. Hisp.-Amer. (4) **6**, 90–91 (1946). (Spanish)

Some results of Hirschman [Duke Math. J. **11**, 793–797 (1944); these Rev. **6**, 127] and Sirvint [Rec. Math. [Mat. Sbornik] N.S. **10**(52), 59–66 (1942); **12**(54), 370–376 (1943); these Rev. **4**, 218; **5**, 262] were anticipated by the author [Madrid thesis, 1936; Revista Unión Mat. Argentina **1**, 71–78 (1937); Revista Acad. Ci. Madrid, 1940].

Loonstra, F. Sur les mouvements presque périodiques. Nederl. Akad. Wetensch., Proc. **49**, 744–751 = Indagationes Math. **8**, 447–454 (1946).

The author gives elementary examples of uniformly almost periodic functions $x=x(t)$, $y=y(t)$ for which the set of points $(x(t),y(t))$, $-\infty<t<\infty$, in the (x,y)-plane is a Jordan arc, or a Jordan curve, or a set of points dense in the interior of a circle, etc. *P. Hartman.*

Nachbin, Leopoldo. Un'estensione di un lemma di Dirichlet. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **3**, 204–208 (1942).

If $f(x)$ is bounded in (a,b) and $\varphi(x)$ is periodic and bounded, with period ω, the integral $\int_a^b f(x)\varphi(nx)dx$ tends,

as $n\to\infty$, to the limit $\omega^{-1}\int_0^\omega \varphi(x)dx\int_a^b f(x)dx$. [This result is known, even under more general assumptions. See, e.g., Zygmund, Trigonometrical Series, Warsaw-Lwów, 1935, p. 173; Mazur and Orlicz, Studia Math. 9, 1–16 (1940); these Rev. 3, 107]. *R. Salem* (Cambridge, Mass.).

Lozinski, S. M. On a theorem of N. Wiener. II. C. R. (Doklady) Acad. Sci. URSS (N.S.) 53, 687–690 (1946).

The author extends to Fourier integrals the well-known theorem of Wiener concerning the relation between the Fourier coefficients of a function f of bounded variation and the jumps of f [J. Math. Phys. Mass. Inst. Tech. 3, 72–94 (1924)]. Thus he gets

$$\lim_{\lambda\to+\infty} 2\lambda\int_{-\infty}^{\infty}|F(u)|^2\sin^2(u/\lambda)du=\sum[f(\xi_k+0)-f(\xi_k-0)]^2,$$

where $f(x)$ is any function of bounded variation, integrable over $(-\infty,+\infty)$, $F(u)=(2\pi)^{-1}\int_{-\infty}^{\infty}f(x)e^{iux}dx$, and $\xi_1,\xi_2,\cdots$ are all the discontinuities of f. The author obtains similar extensions for some of his own results [same C. R. (N.S.) 49, 542–545 (1945); these Rev. 8, 148]. *A. Zygmund.*

Zahorski, Zygmunt. Sur les intégrales singulières. C. R. Acad. Sci. Paris 223, 399–401 (1946).

The author states without proof a number of results concerning the existence of

$$(*)\qquad \lim_{s\to\infty}\int_a^b f(x+t)K(s,t)dt,$$

where f is integrable and the kernel K satisfies certain conditions. The following result may serve as an example. Suppose that $-\infty<a<0<b<+\infty$. There is then a nonnegative kernel $K(s,t)$ satisfying the conditions $\int_a^b K(s,t)dt=1$ for all s,

$$\lim_{s\to\infty}\left\{\int_a^{-\delta}+\int_\delta^b\right\}K(s,t)dt=0$$

for every $\delta>0$, and a bounded function $f(t)$ such that $(*)$ exists at no point x of a set of positive measure.
A. Zygmund (Philadelphia, Pa.).

Gupta, H. C. Two theorems on self-reciprocal functions. J. Indian Math. Soc. (N.S.) 9, 66–68 (1945).

If the operator $E^mE_1^n+E_1^mE^n$, where

$$E(\lambda)=x^{\lambda-1}\int_0^x x^{-\lambda}dx,\quad E_1(\lambda)=-E(1-\lambda),$$

is applied to an R_ν function the resulting function is also R_ν, with a similar theorem for the operators $\Delta=\lambda+xd/dx$, $\Delta_1=-\Delta(1-\lambda)$. *M. C. Gray* (New York, N. Y.).

Gupta, H. C. A theorem on operational calculus. J. Indian Math. Soc. (N.S.) 9, 61–65 (1945).

The theorem of the title is that if $f(p)\doteqdot h(x)$ and $x^\lambda h(x)$ is R_ν, then

$$f(\sqrt{p})\doteqdot\frac{\Gamma(\frac{1}{2}\nu-\frac{1}{2}\lambda+\frac{3}{4})}{2^{\lambda-\frac{1}{2}}\pi^{\frac{1}{2}}\Gamma(\nu+1)}x^{-\frac{1}{2}\lambda-\frac{1}{4}}\int_0^\infty y^{\lambda-\frac{1}{2}}e^{-\frac{1}{2}xy^2}M_{\frac{1}{2}\lambda,\frac{1}{2}\nu}(xy^2)h(y)dy,$$

with suitable restrictions on the form of $h(x)$ to ensure convergence of the integrals. Two integrals involving Bessel and Whittaker functions are evaluated by using the theorem. *M. C. Gray* (New York, N. Y.).

Schwartz, Laurent. Généralisation de la notion de fonction, de dérivation, de transformation de Fourier et applications mathématiques et physiques. Ann. Univ. Grenoble. Sect. Sci. Math. Phys. (N.S.) 21 (1945), 57–74 (1946).

Let μ be an additive set function on $(-\infty,\infty)$, and $\varphi(x)$ an infinitely differentiable function which vanishes outside a finite interval. The author defines

$$\mu(\varphi)=\int_{-\infty}^{\infty}\varphi(x)d\mu,$$

and, in case μ is absolutely continuous,

$$f(\varphi)=\int_{-\infty}^{\infty}\varphi(x)f(x)dx.$$

The "derivatives" $f^{(n)}(\varphi)$ are defined by

$$f^{(n)}(\varphi)=f[(-1)^n\varphi^{(n)}(x)],$$

and the properties of such "functions" studied. By such devices a rigorous treatment can be given not only of the Dirac function $\delta(x)$, but also of the derivatives. It is known of course that $\delta(x)$ can be treated rigorously by an ordinary Stieltjes integral, but apparently an equally satisfactory treatment of the useful functions $\delta^{(n)}(x)$ has proved elusive in the past. All the results are extended to n dimensions, and applications to harmonic functions and Fourier transforms are indicated. *H. Pollard* (Ithaca, N. Y.).

Gilly, Jean. Les parties finies d'intégrales et la transformation de Laplace-Carson. Revue Sci. 83, 259–270 (1945).

This is an amplification of a paper previously reviewed [C. R. Acad. Sci. Paris 218, 100–102 (1944); these Rev. 7, 285]. Most of the material presented is actually well known. Perhaps the two main points of novelty are the following. (a) For an integration range ending with $t=0$ and a singularity of the integrand of type t^{-N-1}, N a positive integer, the author carries out some consequences of using the convention that the logarithmic term is to be dropped. (It is well known that this term is not unique.) The application is to the usual Γ-function relations. (b) The author's previously suggested direct extension of the "finite part" to singularities of the type of essential singularities is now presented in a little more detail in connection with the erf function. The author points out that many of the usual Laplace-Heaviside relations are maintained when the Laplace integrals involve taking finite parts in the usual or extended sense. However, even if the ordinary Laplace transform exists it need not be the same as the transform in the extended sense. This is illustrated by the erf function.
D. G. Bourgin (Urbana, Ill.).

Amerio, Luigi. Sull'inversione della trasformata di Laplace e su alcuni teoremi tauberiani. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) 1, 485–496 (1940).

The author announces results proved in two papers. The first deals with the inversion formula

$$F(t)=\lim_{\lambda\to\infty}(2\pi i)^{-1}\int_{k-i\lambda}^{k+i\lambda}e^{pt}f(p)(1+p^2/\lambda^2)^n dp$$

for the transform $f(p)=\int_{-\infty}^{\infty}e^{-pt}F(t)dt$. The second has already been reviewed [Ann. Mat. Pura Appl. (4) 20, 159–193 (1941); these Rev. 7, 439]. *R. P. Boas, Jr.*

Amerio, Luigi. Sulla trasformata doppia di Laplace. Atti Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. **12**, 707–780 (1942).

Der Verfasser entwickelt eine Theorie der einfachen und vollständigen Doppel-L-Transformation

$$L(F)=L_{\mathrm{I}}(F)=\int_0^\infty\int_0^\infty e^{-px-qy}F(x,y)dxdy=f(p,q),$$

$$L_{\mathrm{II}}(F)=\int_{-\infty}^\infty\int_{-\infty}^\infty e^{-px-qy}F(x,y)dxdy.$$

Die Arbeit untersucht zum Teil gleiche und ähnliche Fragen wie D. L. Bernstein [Duke Math. J. **8**, 460–496 (1941); diese Rev. **3**, 38].

Im ersten Kapitel werden einige einfache Eigenschaften dieser Transformation formuliert und bewiesen. Insbesondere wird der Zusammenhang zwischen den beiden Konvergenzabszissen, die zu den Variabeln p und q gehören, untersucht. Ebenso wird der Abel'sche Grenzwertsatz für Potenz-Reihen auf obige L-Integrale übertragen.

Das zweite Kapitel befasst sich mit den Faltungssätzen. Beispielsweise werden die folgenden Sätze bewiesen. Wenn für (p,q) die Funktion $F_1(x,y)$ in obigem Sinne L-transformierbar ist, das nachstehende Integral

$$\int_0^\infty\int_0^\infty |e^{-px-qy}F_2(x,y)|dxdy$$

konvergiert, dann gilt der Faltungssatz

$$L(F_1*F_2)=L(F_1)\cdot L(F_2).$$

Ein anderer Satz betreffend Faltung längs einer Geraden sagt: Wenn für p, q und den Winkel α, wobei $-\pi<\alpha\leqq\pi$, die Integrale

$$\int_{-\infty}^\infty\int_{-\infty}^\infty |e^{-px-qy}F(x,y)|dxdy,\quad \int_{-\infty}^\infty |e^{(-p\cos\alpha+q\sin\alpha)t}G(t)|dt$$

konvergieren, dann ist

$$G_\alpha*F(x,y)=\int_{-\infty}^\infty G(t)F(x-t\cos\alpha,\ y-t\sin\alpha)dt$$

für fast alle x, y konvergent und es gilt

$$L_{\mathrm{II}}(G_\alpha*F(x,y))=g(p\cos\alpha+q\sin\alpha)L_{\mathrm{II}}(F),$$

wobei

$$g(s)=\int_{-\infty}^\infty e^{-st}G(t)dt.$$

Das dritte Kapitel behandelt die Umkehrung der Doppel-L-Transformation. Bei recht komplizierten Voraussetzungen, wird gezeigt, dass für $F(x,y)$ die folgende Umkehrformel gilt:

$$F(x,y)=(2\pi i)^{-2}\int_{h-i\infty}^{h+i\infty}\int_{k-i\infty}^{k+i\infty} e^{px+qy}f(p,q)dpdq.$$

Insbesondere gilt der Eindeutigkeitssatz, wonach unter der Voraussetzung, dass $f_1(p,q)\equiv f_2(p,q)$ für fast alle (x,y)-Werte geschlossen werden darf: $F_1(x,y)=F_2(x,y)$. Im letzten Kapitel folgen Übertragungen von Sätzen von Lerch, Phragmén und Picone auf obige L-Integrale.

W. Saxer (Zürich).

Sneddon, Ian N. Finite Hankel transforms. Philos. Mag. (7) **37**, 17–25 (1946).

Functional transformations of the type $\int_a^b xf(x)K(\lambda_i x)dx=\bar{f}(\lambda_i)$, where the kernel K is the Bessel function J_n if $a=0$, or a certain linear combination of J_n and the Bessel function of the second kind of order n if $a>0$, are introduced and applied. The arguments λ_i $(i=1,2,\cdots)$ of the transform $\bar{f}$ are the roots of one of the characteristic equations, so that the set of functions $K(\lambda_i x)$ is orthogonal on the interval (a,b). The Fourier-Bessel expansion of $f(x)$ is then an inversion formula for the transformation. The left-hand member of Bessel's differential equation in f transforms into a linear algebraic expression in $\bar{f}$. Thus the partial differential equation in the displacements $z(r,t)$ in a circular membrane, for example, transforms into an ordinary differential equation in $\bar{z}(\lambda_i,t)$; z can be written in terms of $\bar{z}$ with the aid of the inversion formula. The author presents other illustrations. He points out that the method only facilitates the treatment of such problems, since they can be solved also by the generalized Fourier series procedure. *R. V. Churchill.*

Pollard, Harry. Integral transforms. Duke Math. J. **13**, 307–330 (1946).

Suppose $H(s)$ is real, $H(s)\epsilon L(-\infty,\infty)$,

$$\int_{-\infty}^\infty |H(s)|ds<1,\quad H(s)\epsilon L^2(-\infty,\infty),$$

$$h(x)=\int_{-\infty}^\infty e^{-ixs}H(s)ds;$$

in order that λ belong to the point spectrum (to the resolvent set) of the transformation

$$\text{(I)}\qquad f(s)=\int_{-\infty}^\infty H(s-t)g(t)dt$$

it is necessary and sufficient that $\lambda-h(x)$ vanish on a set of positive measure (that $\lambda\neq0$ and $\lambda-h(x)$ vanish nowhere). The author also studies the resolvent, the integral representation and the inversion formula of (I). In the case $h(x)=O(e^{-a|x|})$, $x\to\infty$, for some positive value of a, the inversion formula for (I) can be written

$$g(s)=\underset{n\to\infty}{\mathrm{l.i.m.}}\ p_n\left(\frac{1}{i}\frac{d}{ds}\right)f(s),$$

where $p_n(x)$ is a sequence of polynomials such that $p_n(x)h(x)$ converges to unity boundedly almost everywhere.

These results are applied to the transformation of Stieltjes $(H(s)=(2\pi)^{-1}\operatorname{sech}\frac{1}{2}(s-t))$ and to other particular cases. Similar results are established for the transformation $f(s)=\int_{-\infty}^\infty H(s+t)g(t)dt$ and applied to the Laplace transformation. *C. Miranda* (Naples).

Geronimus, J. On the trigonometric moment problem. Ann. of Math. (2) **47**, 742–761 (1946).

Let $\sigma(\theta)$ be a nondecreasing function with infinitely many points of increase, $P_n(z)=z^n+\cdots$ the associated orthogonal polynomials in $[0,2\pi]$, so that

$$\int_{-\pi}^\pi P_n(z)\overline{P_m(z)}d\sigma(\theta)=0,\qquad z=e^{i\theta},\ n\neq m.$$

Putting $a_n=-\bar{P}_{n+1}(0)$ we have [see Szegö, Orthogonal Polynomials, Amer. Math. Soc. Colloquium Publ., v. 23, New York, 1929, p. 286; these Rev. **1**, 14]

$$P_{n+1}(z)=zP_n(z)-\bar{a}_nP_n^*(z).$$

The author also considers the polynomials defined by

$$\Omega_{n+1}(z)=z\Omega_n(z)+\bar{a}_n\Omega_n^*(z),\qquad n=0,1,\cdots;\ \Omega_0=1,$$

and studies various properties of P_n and Ω_n. Asymptotic formulas for the polynomials $P_n(z)$ are obtained as $n\to\infty$,

under the conditions $\sum|a_n|^2<\infty$ and $\sum|a_n|<\infty$. In the second case we have

$$P_n(z)=z^n\{\bar{\pi}(\bar{z}^{-1})+\epsilon_n\}, \qquad |z|\geqq 1,$$

where π is a certain analytic function associated with $\sigma(\theta)$. The error ϵ_n is in this case $O(\sum_{k=n}^{\infty}|a_k|)$. According to a result of the reviewer these theorems can be applied to polynomials orthogonal on a finite part of the real axis.
G. Szegö (Stanford University, Calif.).

Polynomials, Polynomial Approximations

Iglisch, Rudolf. Zur Stetigkeit der Wurzeln einer algebraischen Gleichung. Deutsche Math. 7, 520–521 (1944).

An elementary proof is given for the following theorem of H. Kneser [Math. Z. 48, 101–104 (1942); see these Rev. 4, 273]. Given any polynomial $f(x)=x^n+a_1x^{n-1}+\cdots+a_n$ and a positive number δ, one can find a positive ϵ independent of the a_j such that, if the derivatives $|f^{(\mu)}(c)|\leqq\epsilon$ for $\mu=0, 1, \cdots, m-1$, then at least m zeros of $f(x)$ lie in the circle $K:|x-c|\leqq\delta$. The new proof is constructed by induction on m. Obviously true for $m=1$, the theorem is assumed true for $m-1$. If it is false for m, then a sequence of nth degree polynomials $f_k(x)$ having precisely $m-1$ zeros $x_{k,p}$ in K can be selected so that $|f_k^{(\mu)}(c)|\leqq\epsilon_k$ for $\mu=0, 1, \cdots, m-1$ with $\epsilon_k\to 0$ as $k\to\infty$. On writing $f_k(x)=(x-x_{k,1})\cdots(x-x_{k,m-1})g_k(x)$ with $g_k(x)\neq 0$ in K, one can prove that, as $k\to\infty$, $f_k^{(m-1)}(x)/g_k(x)\to(m-1)!$, in contradiction to the assumption $|f_k^{(m-1)}(c)|<\epsilon_k$.
M. Marden.

Delange, H. Sur les suites de polynomes ou de fonctions entières à zéros réels. Ann. Sci. École Norm. Sup. (3) 62, 115–183 (1945).

It is impossible to give a complete picture of the content of this paper in a short review. Only the principal points can be mentioned. In an earlier paper [same Ann. (3) 56, 173–275 (1939); these Rev. 1, 310] the author has studied sequences of polynomials $\{P_n(z)\}$, in particular, the relation of convergence properties to the distribution of zeros. In the present paper it is assumed that all zeros of $P_n(z)$ are real and less than a fixed constant a. We denote by $\nu_n(t)$ the number of zeros of $P_n(z)$ which are not less than t. Then the main result can be formulated as follows. Suppose $\varphi(n)$ is a positive function and the complex constants c_n are chosen in such a way that $(\varphi(n))^{-1}\log|c_nP_n(z)|$ converges towards a certain function in an arbitrarily small domain or on a real interval to the right of a. Then $\lim(\varphi(n))^{-1}\nu_n(t)=\nu(t)$ exists at every point where $\nu(t)$ is continuous. Also, if $A>a$,

$$\lim_{n\to\infty}(\varphi(n))^{-1}P_n'(A)/P_n(A)$$

exists. This theorem is applied to a sequence of polynomials defined by a recurrence formula of the type

$$P_{n+1}(z)-(a_nz+b_n)P_n(z)+c_nP_{n-1}(z)=0,$$

where a_n, b_n, c_n satisfy certain conditions. Special cases, containing Laguerre polynomials, are also considered.

The further part of the paper deals with entire functions of a fixed genus, the zeros being subject to the same conditions as above. Also entire functions of finite genus are considered whose zeros have the property that their arguments approach a limit when the zeros tend to infinity. In both cases relations are established between the distribution of zeros and the growth of the modulus of the function in a properly chosen direction.
G. Szegö.

Turán, P. On rational polynomials. Acta Univ. Szeged. Sect. Sci. Math. 11, 106–113 (1946).

Let $f(x)$ be a polynomial of degree n with real coefficients such that $f(\pm 1)=0$ and $f(x)\neq 0$ for $-1<x<1$. If the absolute maximum of $f(x)$ in $(-1, 1)$ is attained for $x=\xi$ we have

$$|\xi|\leqq\cos(\pi/n), \qquad n \text{ even},$$

$$|\xi|\leqq\frac{3\cos(\pi/n)-1}{1+\cos(\pi/n)}, \qquad n \text{ odd}.$$

These bounds are the best possible. The following theorem is also proved. Let $G(z)$ be a polynomial of degree n and $\max|G(z)|=|G(1)|$ for $|z|\leqq 1$. Such a polynomial cannot vanish on the open arc $|\text{arc }z|<\pi/n$ of the unit circle; it vanishes at the end-points of this arc only if $G(z)=c(1+z^n)$.
G. Szegö (Stanford University, Calif.).

Mazurkiewicz, Stefan. Un théorème sur les polynômes. Ann. Soc. Polon. Math. 18, 113–117 (1945).

The author proves the following theorem, which can also be expressed in terms of the concept of the transfinite diameter. To each $\epsilon>0$ an $\eta>0$ corresponds such that the following is true. If C is an arbitrary continuum of diameter 1 and E a closed set, $E\subset C$, such that the linear measure of $C-E$ is less than η, then for an arbitrary polynomial $P(z)$ of degree n we have

$$\max_{z\in C}|P(z)|<(1+\epsilon)^n\max_{z\in E}|P(z)|.$$

G. Szegö (Stanford University, Calif.).

Popoviciu, Tiberiu. Sur l'approximation des fonctions continues d'une variable réelle par des polynomes. Ann. Sci. Univ. Jassy. Sect. I. 28, 208 (1942).

A simple derivation of the author's earlier result [Mathematica, Cluj 10, 49–54 (1935)] concerning the order of approximation of continuous functions by Bernstein polynomials. [The present proof is identical, except for avoiding probabilistic terminology, with the reviewer's proof in Studia Math. 8, 170 (1939). The same proof was also found by Natanson, C. R. (Doklady) Acad. Sci. URSS (N.S.) 45, 274–277 (1944); these Rev. 6, 267.]
M. Kac.

Nikolsky, S. Sur la meilleure approximation au moyen des polynômes des fonctions vérifiant la condition de Lipschitz. C. R. (Doklady) Acad. Sci. URSS (N.S.) 52, 7–9 (1946).

Soit (MH) la classe des fonctions continues, définies pour $-1\leqq x\leqq 1$, et satisfaisant à la condition de Lipschitz $|f(x'')-f(x')|\leqq M|x''-x'|$ et E_nf la meilleure approximation de f par des polynomes en x de degré $n-1$ au plus. En réponse à un problème posé par le rapporteur [Bull. Sci. Math. (2) 62, 338–351 (1938), en particulier, p. 344] l'auteur énonce des résultats tels que les suivants:

(1) $$\sup_{f\in(MH)}E_nf=\tfrac{1}{2}M\pi/n-\epsilon_n, \qquad 0<\epsilon_n=O(1/(n\log n));$$

(2) il y a dans (MH) des fonctions telles que $\lim\sup_{n\to\infty}nE_nf=\frac{1}{2}M\pi$; (3) une méthode de sommation asymptotiquement la meilleure pour les fonctions de (MH) consiste à prendre le développement de f en série de polynomes trigonométriques et à lui appliquer la méthode de sommation introduite par le rapporteur pour les fonctions périodiques satisfaisant à la conditon de Lipschitz ci-dessus. Pour les sommes partielles de ce développement l'auteur résout un problème analogue à celui posé par Kolmogoroff [Ann. of Math. (2) 36, 521–526 (1935)].
J. Favard (Paris).

Merli, L. Sulla convergenza in media della formula di interpolazione di Lagrange. Giorn. Ist. Ital. Attuari 12, 34–42 (1941).

Merli, L. Sulla convergenza in media della formula di interpolazione di Hermite per un particolare sistema di punti interpolanti. Giorn. Ist. Ital. Attuari 12, 221–226 (1941).

Soit $f(x)$ une fonction continue pour $-1\leqq x\leqq 1$; posant $x=\cos\theta$, l'auteur considère les polynomes

$$w_n(x)=\sin (n+\tfrac{1}{2})\theta/\sin \tfrac{1}{2}\theta$$

et démontre les résultats suivants. (1) Soit $L_n(x)$ le polynome interpolateur de $f(x)$ relativement aux zéros de $w_n(x)$; alors, si $f(x)$ satisfait à une condition de Lipschitz d'ordre $\alpha>\frac{1}{2}$,

$$\lim_{n\to\infty}\int_{-1}^{1}\{f(x)-L_n(x)\}^2dx=0.$$

(2) Si $f(x)$ a une dérivée continue satisfaisant à une condition de Lipschitz d'ordre $\alpha+\frac{1}{2}$, en désignant par $S_n(x)$ le polynome (de degré $2n-1$ au plus) qui, aux zéros de $w_n(x)$, prend les mêmes valeurs que $f(x)$, tandis que $S_n'(x)$ y prend les mêmes valeurs que $f'(x)$, on a

$$\lim_{n\to\infty}\int_{-1}^{1}\{f(x)-S_n(x)\}^2dx=0.$$

J. Favard (Paris).

Merli, L. Sulla convergenza in media della derivata del polinomio interpolante di Lagrange. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 1, 322–325 (1946).

Let $f(x)$ be a differentiable function defined in $(-1, +1)$ with $f'(x)\varepsilon\mathrm{Lip}\,\alpha$, $\alpha>\frac{1}{2}$. Let $L_n(x)$ be the Lagrange interpolating polynomial coinciding with $f(x)$ at the points $x_k=\cos\theta_k$, where $\theta_k=(2k-1)\pi/2n$ $(k=1, 2, \cdots, n)$. The author shows that then

$$\int_{-1}^{1}(1-x^2)^{\frac{1}{2}}\{f'(x)-L_n'(x)\}^2dx\to 0.$$

[For trigonometric interpolation with fundamental points θ_k, much stronger results are known to be true. See J. Marcinkiewicz, Studia Math. 6, 1–17, 67–81 (1936).]

A. Zygmund (Philadelphia, Pa.).

v. Szász, Paul. Über die äquidistante Interpolation. Mat. Fiz. Lapok 49, 63–69 (1942). (Hungarian. German summary)

Let $f(x)$ be of bounded variation in (a, b) and continuous at $\frac{1}{2}(a+b)$. Denote by $L_n(x)$ the unique polynomial of degree not exceeding $n-1$, with

$$L_n(f(a+(i-1)(b-a)/(n-1)) \\ =f(a+(i-1)(b-a)/(n-1)),\quad i=1, \cdots, n.$$

The author proves that $L_n(\frac{1}{2}(a+b))\to f(\frac{1}{2}(a+b))$.

P. Erdös (Syracuse, N. Y.).

Grünwald, Géza. Über die Grundfunktionen der Interpolation. Mat. Fiz. Lapok 49, 76–83 (1942). (Hungarian. German summary)

The author shows that for normal point groups $\lim_{n\to\infty}\sum_{k=1}^{n}(l_k^{(n)}(x))^2=1$ for all x satisfying $-1<x<1$. A paper of the author containing this and other results has already been reviewed [Acta Math. 75, 219–245 (1943); these Rev. 7, 157].

P. Erdös (Syracuse, N. Y.).

Leja, F. Sur un problème de l'interpolation. Ann. Soc. Polon. Math. 18, 123–128 (1945).

Soit $f(x)$ une fonction continue pour $0\leqq x\leqq 1$; pour toute valeur de n $(=1, 2, \cdots)$ on donne: (1) $n+1$ nombres $\xi_{k,n}$ $(0\leqq\xi_{k,n}\leqq 1,\ \xi_{k,n}<\xi_{k+1,n})$, la distance maxima entre deux de ces points tendant vers zéro lorsque n augmente indéfiniment, (2) $n+1$ fonctions continues $P_{k,n}(x)$, et on considère la suite de fonctions $\pi_n(x)=\sum_{k=0}^{n}f(\xi_{k,n})P_{k,n}(x)$. La suite $\pi_n(x)$ converge uniformément vers $f(x)$ si: (1) $\lim_{n\to\infty}\sum_{k=0}^{n}P_{k,n}(x)=1$ (uniformément); (2) quel que soit l'intervalle fermé $[\alpha, \beta]$ dans $[0, 1]$, $\sum_{[\alpha,\beta]}|P_{k,n}(x)|$ étendue à tous les indices k pour lesquels $\xi_{k,n}$ appartient à $[\alpha, \beta]$ tend uniformément vers zéro dans tout intervalle partiel de $[0, 1]$ extérieur à $[\alpha, \beta]$; (3) il existe un nombre M, indépendant de n, tel que $\sum_{k=0}^{n}|P_{k,n}(x)|<M$. Ce résultat a déjà été obtenue par le rapporteur [J. Math. Pures Appl. (9) 23, 219–247 (1944), en particulier, pp. 223–224; ces Rev. 7, 436]. L'auteur montre ensuite que la méthode d'interpolation de Lagrange ne satisfait pas à l'hypothèse (3).

J. Favard (Paris).

Bernstein, S. Sur la meilleure approximation des fonctions $\int_0^\infty|y|^sd\psi(s)$ sur le segment $(-1, +1)$. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 10, 185–196 (1946). (Russian. French summary)

Let $f(y)=\int_0^\infty|y|^sd\psi(s)$, $f_A(y)=\int_0^A|y|^sd\psi(s)$ and suppose that $\int_0^A su^{2s}|d\psi(s)|=O((\log|u|)^{-1-\epsilon})$ for sufficiently large A. Let $E_n(f(y))$ denote the best approximation to $f(y)$ on $-1\leqq y\leqq 1$ by polynomials of order n, that is, the maximum of $|f(y)-P_n(y)|$ for $-1\leqq y\leqq 1$, when $P_n(y)$ is the polynomial of order n for which this maximum is least. Suppose that, for some number $A>0$, $\lim_{n\to\infty}E_n(f(y))/E_n(f_A(y))=1$ and that $E_n(f(y))>\exp(-n^\rho)$. The author finds an asymptotic formula for the difference between $f(y)$ and the interpolating polynomial of even degree $m=2n$ which is equal to $f(y)$ at k Chebyshev abscissae y_k, where $\cos(m \arccos y_k)=0$; $k=0, 1, \cdots, m-1$; $y_m=0$.

This result is applied to the particular function

$$2\int_a^\infty|y|^{2s}e^{2b(a-s)}ds=|y|^{2a}/(b-\log|y|),$$

where $b>0$, $a\geqq 0$ and $|y|<e^b$.

A. C. Offord.

Sansone, G. Separazione degli zeri del polinomio di Jacobi $P_n^{(0,-1)}(x)=\frac{1}{2}[P_n(x)+P_{n-1}(x)]$. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) 10(79), 9 pp. (1946).

Let ω_i be the zeros of $P_n(\cos\omega)+P_{n-1}(\cos\omega)$, $0<\omega_1<\omega_2<\cdots<\omega_{n-1}<\omega_n=\pi$. Then

$$\pi(i-\tfrac{1}{4})/(n+\tfrac{1}{2})<\omega_i<\pi i/n,\qquad i=1, 2, \cdots, [n/2],$$
$$\pi i/(n+1)<\omega_i<\pi(i-\tfrac{1}{4})/(n-\tfrac{1}{2}),\qquad i=[n/2]+1, \cdots, [(5n-1)/6],$$
$$\pi k/(n+\tfrac{1}{8}n^{-1})<\pi-\omega_{n-k}<\pi(k+\tfrac{1}{4})/(n-\tfrac{1}{8}n^{-1}),\qquad n\geqq 3,\ k=1, 2, \cdots, [(n-1)/3].$$

G. Szegö (Stanford University, Calif.).

Feldheim, Ervin. Contributions à la théorie des polynomes de Jacobi. Mat. Fiz. Lapok 48, 453–504 (1941). (Hungarian. French summary)

Various new properties of the Jacobi polynomials defined by

$$P_n^{(\alpha,\beta)}(x)=\frac{(\alpha-1)_n}{n!}F\left(n+\alpha+\beta+1, -n; \alpha+1; \frac{1-x}{2}\right),$$

$$(a)_n=\frac{\Gamma(a+n)}{\Gamma(a)},$$

are studied. It is impossible to enumerate all the results; we shall mention only the most conspicuous details. (a) The following new generating series is derived:

$$\sum_{n=0}^{\infty} \frac{(\alpha+\beta+1)_n}{(\alpha+1)_n} P_n^{(\alpha,\beta)}(x)t^n = (1-t)^{-\alpha-\beta-1} F\left(\frac{\alpha+\beta+1}{2}, \frac{\alpha+\beta}{2}+1; \alpha+1; -\frac{2t(1-x)}{(1-t)^2}\right), \quad |t|<1.$$

(b) Other generating series, in which α or β depend on n, are also studied. (c) Generalizing results of Watson, generating series for the product $P_n^{(\alpha,\beta)}(x)P_n^{(\alpha,\beta)}(y)$ are established. (d) The expression $P_n^{(\alpha,\beta)}\{1-\mu(1-x)\}$ is represented in terms of $P_\nu^{(\gamma,\delta)}(x)$, the coefficients being expressible in terms of functions of μ of the type ${}_3F_2$. (e) The product $P_m^{(\alpha,\beta)}(x)P_n^{(\gamma,\delta)}(x)$ is expressed in terms of $P_\nu^{(\kappa,\sigma)}(x)$. (f) Integral representations are established. (g) By appropriate limiting processes results on Laguerre and Hermite polynomials are obtained. Finally generalizations of various kinds are investigated. In all these instances ultraspherical polynomials and various special cases of them are used as examples or illustrations. *G. Szegö.*

Tricomi, F. Sviluppo dei polinomi di Laguerre e di Hermite in serie di funzioni di Bessel. Giorn. Ist. Ital. Attuari 12, 14–33 (1941).

In this paper the author generalizes an asymptotic formula for the Laguerre polynomials $L_n^{(0)}(t)$ first given by E. Moecklin [Comment. Math. Helv. 7, 24–46 (1934)] and obtains the result

$$(1)\qquad L_n^{(\alpha)}(t) = e^{ht}\frac{\Gamma(\alpha+n+1)}{n!n^\alpha}\sum_{\nu=0}^{\infty} A_\nu(h)(t/n)^{(\nu-\alpha)/2}J_{\alpha+\nu}(2(nt)^{\frac{1}{2}}),$$

where h is an arbitrary nonnegative constant and $A_\nu(h)$ is the coefficient of z^ν in the power series expansion of

$$(2)\qquad f(z) = e^{nz}\{1+(h-1)z\}^n/(1+hz)^{\alpha+n+1},$$

and satisfies the four-term recurrence relation

$$(3)\qquad (\nu+1)A_{\nu+1} = \{(1-2h)\nu-(\alpha+1)h\}A_\nu - \{(1-2h)n+h(h-1)(\alpha+\nu)\}A_{\nu-1}+h(h-1)nA_\nu,$$

$\nu=2, 3, 4, \cdots$, with the initial values

$$A_0(h)=1,\quad A_1(h)=-(\alpha+1)h,\quad A_2(h)=\binom{\alpha+2}{2}h^2+(h-\tfrac{1}{2})n.$$

It is furthermore shown that the series (1) converges absolutely and uniformly in the interval $0\leqq t<n/h^2$. A discussion of the coefficients $A_\nu(h)$ leads to the result that they are expressible in terms of the Laguerre polynomials as follows: if $h=0$, $A_\nu(0)=(-1)^\nu L_\nu^{(n-\nu)}(n)$; if $h\neq 0$,

$$A_\nu(h) = h^{\nu-n}\sum_{k=0}^{n}\binom{n}{k}(h-1)^{n-k}L_\nu^{-(\alpha+\nu+k+1)}(-n/h).$$

Using well-known formulas connecting Hermite and Laguerre polynomials, the author extends his results to the Hermite polynomials, obtaining

$$H_{2n}(x) = (-1)^n 1\cdot 3\cdot \cdots \cdot(2n-1)x^2e^{\frac{1}{2}hx^2} \times\sum_{\nu=0}^{\infty} A_\nu^{(-\frac{1}{2})}(h)(2n)^{1-\nu}G_{\nu-1}(x(2n)^{\frac{1}{2}}),$$

$$H_{2n+1}(x) = (-1)^n 1\cdot 3\cdot \cdots \cdot(2n+1)xe^{\frac{1}{2}hx^2} \times\sum_{\nu=0}^{\infty} A_\nu^{(\frac{1}{2})}(h)(2n)^{-\nu}G_\nu(x(2n)^{\frac{1}{2}}),$$

in which the coefficients $A_\nu^{(-\frac{1}{2})}(h)$, $A_\nu^{(\frac{1}{2})}(h)$ satisfy (3) with $\alpha=-\frac{1}{2}$ and $\alpha=\frac{1}{2}$, respectively, and

$$G_\nu(\xi) = (\tfrac{1}{2}\pi\xi)^{\frac{1}{2}}\xi^{\nu-1}J_{\nu+\frac{1}{2}}(\xi),\qquad \nu=0, 1, 2, \cdots,$$

$$G_{-1}(\xi) = \xi^{-2}(\tfrac{1}{2}\pi\xi)^{\frac{1}{2}}J_{-\frac{1}{2}}(\xi) = \xi^{-2}\cos\xi.$$

The author also investigates the coefficients $A_\nu(h)$ as polynomials in n. These polynomials are in general of degree $[\nu/2]$ and it is shown that for $h=\frac{1}{2}$ the coefficient of the highest power of n vanishes. The importance of this result is that for this choice of h the convergence of the series is more rapid. *M. A. Basoco* (Lincoln, Neb.).

Bordoni, Piero Giorgio. Sulle funzioni di Stokes. Ricerca Sci. 15, 149–151 (1945).

In the investigation of the simple wave equation in spherical coordinates, one procedure obtains as one element in the general solution the Stokes function of the first kind, $F(-n, n+1; -\frac{1}{2}jkr)$. These polynomials, which are a special case of the well-known Laguerre polynomials, are discussed briefly with certain recurrence equations and relations to Bessel functions indicated. *N. A. Hall.*

Special Functions

Skolem, S. Some definite integrals of the form

$$\int_0^\infty f(x)\cos\alpha x\,dx \quad\text{and}\quad \int_0^\infty f(x)\sin\alpha x\,dx.$$

Norsk Mat. Tidsskr. 27, 65–75 (1945). (Norwegian)

The author obtains

$$S(\alpha) = \int_0^\infty (1+x^2)^{-1}\sin\alpha x dx = \tfrac{1}{2}\{e^{-\alpha}\,\overline{\mathrm{Ei}}\,(\alpha) - e^\alpha\,\mathrm{Ei}\,(-\alpha)\},$$

where the notation for exponential integrals is that of Jahnke and Emde's Tables. He tabulates $S(\alpha)$ and $S'(\alpha)$ for $\alpha=0(.01).1(.1)1(1)10(10)100$, to 5 decimals. He expresses some other integrals in terms of $S(\alpha)$ and $C(\alpha)$, where $C(\alpha)$ is obtained from $S(\alpha)$ by replacing sin by cos ($C(\alpha)=\frac{1}{2}\pi e^{-\alpha}$). *R. P. Boas, Jr.* (Providence, R. I.).

Picone, Mauro. Complementi analitici e numerici ad una ricerca di Signorini sul moto di un sistema soggetto a resistenza idraulica e forza di richiamo. Ist. Veneto Sci. Lett. Arti. Parte II. Cl. Sci. Mat. Nat. 101, 473–492 (1942).

The equation $\ddot{q}+2k|\dot{q}|\dot{q}+w^2q=0$ occurs in hydraulic theory, and it is of importance to determine the constant k from numerical data furnished by experiment. In this connection the integral

$$z(x) = \int_{-x}^{y(x)} \frac{e^{s/2}ds}{\{(1-s)e^s-(1+x)e^{-x}\}^{\frac{1}{2}}}$$

arises, where $(1-y)e^y=(1+x)e^{-x}$, $y>0$. The author discusses the behavior of $z(x)$ as a function of x and derives results for the cases $x\to 0$, $x\to 1$, $x\to\infty$. *R. Bellman.*

Sansone, G. Sulla durata delle oscillazioni di un punto soggetto a resistenza idraulica e forza di richiamo. Valutazione asintotica. Ist. Veneto Sci. Lett. Arti. Parte II. Cl. Sci. Mat. Nat. 102, 53–72 (1943).

A more detailed discussion of the integral mentioned in the preceding review. The author obtains asymptotic estimates for $z(x)$ and $z'(x)$ as $x\to\infty$, in addition to results for the cases $x\to 0$, $x\to 1$. *R. Bellman.*

Pollard, Harry. The representation of e^{-s^λ} as a Laplace integral. Bull. Amer. Math. Soc. 52, 908–910 (1946).

This paper is concerned with the inversion of the integral

$$e^{-s^\lambda}=\int_0^\infty e^{-st}\phi_\lambda(t)dt, \qquad 0<\lambda<1.$$

The author combines theorems of Bochner [Duke Math. J. 3, 488–502 (1937)] and Hille and Tamarkin [Proc. Nat. Acad. Sci. U. S. A. 19, 902–908 (1933)] to show that the function $\phi_\lambda(t)$ is positive almost everywhere and that $\int_0^\infty\phi_\lambda(t)dt<\infty$. An application of the Post-Widder inversion theorem [D. V. Widder, The Laplace Transform, Princeton University Press, 1941; these Rev. 3, 232] and of a theorem of Post [Trans. Amer. Math. Soc. 32, 723–781 (1930)] shows that

$$\phi_\lambda(t)=(2\pi i)^{-1}\int_\gamma e^{st}e^{-s^\lambda}ds$$

is valid for all $t>0$, where γ is the contour $x/a+|y|/b=1$, a, b fixed and positive. The rest of the paper is concerned with the evaluation of this contour integral; it is shown that

$$\phi_\lambda(t)=\pi^{-1}\int_0^\infty e^{-tu}e^{-u^\lambda\cos\pi\lambda}\sin(u^\lambda\sin\pi\lambda)du$$
$$=-\pi^{-1}\sum_{k=0}^\infty\frac{(-1)^k\Gamma(\lambda k+1)\sin\pi\lambda k}{k!t^{\lambda k+1}}.$$

This result was obtained formally by P. Humbert [Bull. Sci. Math. (2) 69, 121–129 (1945); these Rev. 7, 439]. In particular, the following special cases are noted:

$$\phi_{\frac12}(t)=\tfrac12\pi^{-\frac12}t^{-\frac32}e^{-1/(4t)},$$
$$\phi_{\frac13}(t)=-\tfrac13(3\pi)^{-\frac12}t^{-1}e^{-2/27t^2}W_{-\frac12,-1/6}(-4/27t^2),$$

where $W_{m,n}(z)$ is the Whittaker function.

M. A. Basoco (Lincoln, Neb.).

Selberg, Atle. Remarks on a multiple integral. Norsk Mat. Tidsskr. 26, 71–78 (1944). (Norwegian)

In this paper the following result is derived:

$$\int_0^1\cdots\int_0^1(u_1\cdots u_p)^{x-1}\{(1-u_1)\cdots(1-u_p)\}^{y-1}$$
$$\times|\Delta(u)|^{2z}du_1\cdots du_p$$
$$=\prod_{\nu=1}^p\frac{\Gamma(1+\nu z)\Gamma\{x+(\nu-1)z\}\Gamma\{y+(\nu-1)z\}}{\Gamma(1+z)\Gamma\{x+y+(p+\nu-2)z\}};$$

p is a positive integer; $\Re x>0$, $\Re y>0$, $\Re z$ is greater than the largest of $-p^{-1}$, $-\Re x/(p-1)$, $-\Re y/(p-1)$; $\Delta(u)$ is the discriminant $\prod_{i<j}(u_j-u_i)$. This result is initially derived for z a positive integer and afterwards extended by analytic continuation to complex z. The main point in this investigation is the evaluation of

$$\int_0^1\cdots\int_0^1|\Delta(u)|^{2z}du_1\cdots du_p$$
$$=\frac{1}{\{\Gamma(1+z)\}^p}\prod_{\nu=1}^p\frac{\Gamma(1+\nu z)\Gamma^2\{1+(\nu-1)z\}}{\Gamma\{2+(p+\nu-2)z\}},$$

for z a nonnegative integer.

S. C. van Veen.

Strachey, C., and Wallis, P. J. Hahn's functions $S_m(\alpha)$ and $U_m(\alpha)$. Philos. Mag. (7) 37, 87–94 (1946).

The functions in question are

$$-S_m(\alpha)=\sum_{n=1}^\infty\frac{m^2\sin^2 n\pi\alpha}{n(m^2-n^2\alpha^2)},\quad U_m(\alpha)=\sum_{n=1}^\infty\frac{m^2n\sin^2 n\pi\alpha}{\alpha^2(n^2-m^2/\alpha^2)^2},$$

with $0<\alpha<1$. In this paper closed expressions are derived for the case of rational α and integral m. These closed expressions depend on $\psi(x)$, the logarithmic derivative of the factorial function. The closed expressions are used to compute tables to five decimals of $-S_m(\alpha)$ for $m=1, 2, \cdots, 10$; $\alpha=0, 0.1, 0.2, \cdots, 1.0, \frac14, \frac34, \frac13, \frac23$ and of $m^{-1}U_m(\alpha)$ for $m=1, 2, \cdots, 10$; $\alpha=0, \frac14, \frac12, \frac34, 1, \frac13, \frac23$. There are also integral representations for integral m, power series expansions in powers of α and asymptotic expansions for large m. The authors seem unacquainted with the recent results of L. S. Goddard [Proc. Cambridge Philos. Soc. 41, 145–160 (1945); these Rev. 7, 66].

A. Erdélyi (Edinburgh).

Lebedev, N. N. Équations intégrales pour les solutions périodiques de l'équation

$$u''+(a_0+a_1\cos 2x+a_2\cos 4x)u=0.$$

C. R. (Doklady) Acad. Sci. URSS (N.S.) 52, 391–394 (1946).

The differential equation of the title admits the following periodic solutions: (1) even solutions of period π, (2) even solutions of period 2π, (3) odd solutions of period π, (4) odd solutions of period 2π. These solutions are studied in connection with those of the integral equation

$$u(x)=\lambda\int_0^{\pi/2}K(x,s)u(s)ds,$$

in which the kernel K is of the type

$$K(x,s)=\varphi(x)\varphi(s)F(\cos 2x+\cos 2s).$$

In particular, if $\varphi(x)$ is one of the functions 1, $\cos x$, $\sin x$, $\sin 2x$, and $F(t)$ satisfies

$$F''+t^{-1}(2m+1)F'-(\tfrac18a_2+\tfrac14a_1/t)F=0,$$

with $m=0, \frac12, \frac12, 1$, respectively, then the solutions of the integral equations are those of the differential equation as specified above, provided F is expressed in terms of hypergeometric functions,

$$F(t)=\exp(-t(a_2/2)^{\frac12})\,{}_1F_1(\tfrac12+m+\tfrac14a_1(2a_2)^{-\frac12},\ 2m+1,\ t(2a_2)^{\frac12}).$$

It is shown that either $a_1=0$ or $a_2=0$ leads to known results for Mathieu functions. For $a_1=\pm(p+m+\frac12)4(2a_2)^{\frac12}$, in which p is any nonnegative integer, previous results of Whittaker [Proc. Edinburgh Math. Soc. 33, 14–23 (1915)] and Ince [Proc. London Math. Soc. (2) 23, 6–74 (1925), in particular, p. 55] are obtained.

[Reviewer's note: The author is wrong in stating that, except when $a_1=a_2=0$, the differential equation does not admit two independent periodic solutions for any given set a_0, a_1, a_2. From general theory even two independent periodic solutions of period π may sometimes exist; this is the case for the equation under consideration, as is evident from numerical work of Klotter and Kotowski, Z. Angew. Math. Mech. 23, 149–155 (1943); these Rev. 5, 203.]

C. J. Bouwkamp (Eindhoven).

Basu, K. A few applications of some well-known transforms to functions involving Sonine's polynomials. Proc. Nat. Acad. Sci. India. Sect. A. 14, 181–190 (1945).

Most of the results of this note are known and the proofs are not essentially different from the usual ones.

G. Szegö (Stanford University, Calif.).

Mursi, M. On the solution of linear differential equations by definite integrals. Proc. Math. Phys. Soc. Egypt 3 (1945), 13–18 (1946). (English. Arabic summary)

Operational methods are used to obtain representations of the hypergeometric functions of two variables [Appell,

J. Math. Pures Appl. (3) 8, 173–216 (1882)] in terms of double integrals involving gamma functions. These results may be obtained from the representation of the hypergeometric function of one variable in terms of Barnes's integrals. *F. G. Dressel* (Durham, N. C.).

Andreoli, Giulio. Equazioni differenziali e funzioni metageometriche. Rend. Accad. Sci. Fis. Mat. Napoli (4) 12, 316–323 (1942).

The "hypergeometric function of order r" ($H.F_r$) is defined here by

$$G_r(x)=G\begin{pmatrix} a_1, h_1; \cdots; a_r, h_r \\ b_1, k_1; \cdots; b_r, k_r \end{pmatrix}; x\Big)=\sum_{n=0}^{\infty}\Omega_n^{(r)}x^n$$

with

$$\Omega_{n+1}^{(r)}=\Omega_n^{(r)}\frac{(a_1+nh_1)\cdots(a_r+nh_r)}{(b_1+nk_1)\cdots(b_r+nk_r)}, \qquad \Omega_0^{(r)}=1.$$

A $H.F_r$ is called regular if all h, k are different from 0, special (ρ) if $\rho(\leqq r)$ of the quantities h are 0, singular (ρ) if $\rho(\leqq r)$ of the quantities k are 0. The author derives the known results that a $H.F_r$ satisfies a linear differential equation of order r with coefficients of the form $x^{r-m}(u_m x+v_m)$, $m=0, \cdots, r$, where u_m, v_m are rational functions of a, b, h, k, and that a regular $H.F_r$ is derived from a $H.F_{r-1}$ by an integral of the form

$$G_r(x)=\int_0^{\rho} Z^{\lambda}(\rho-Z)^{\mu}G_{r-1}(Zx)dZ$$

for certain definite values of ρ, λ, μ. *S. C. van Veen.*

Harmonic Functions, Potential Theory

Mineo, Corradino. Su una formula integro-differenziale relativa alle funzioni di Laplace. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) 2, 944–947 (1941).

This note is concerned with the detailed demonstration of the formula

$$\int_\sigma\left\{Y_n^{(0)}-Y_n+(\sin\theta\cos\theta_0\cos(\phi-\phi_0)-\cos\theta\cos\theta_0)\frac{\partial Y_n^{(0)}}{\partial\theta_0}\right.$$
$$\left.+\sin\theta\sin(\phi-\phi_0)\csc\theta_0\frac{\partial Y_n^{(0)}}{\partial\phi_0}\right\}\csc^3\tfrac{1}{2}\gamma d\sigma_n=16\pi n Y_n^{(0)},$$

where $Y_n\equiv Y_n(\theta,\phi)$ is a surface spherical harmonic of order n, $Y_n^{(0)}=Y_n(\theta_0,\phi_0)$, $d\sigma_n=\sin\theta d\theta d\phi$, σ is a spherical surface of unit radius and γ is the arc PM, PP_0M being a spherical triangle with vertices $P(\theta_0,\phi_0)$, $M(\theta,\phi)$ and P_0 the pole $(0, 0)$. *M. A. Basoco* (Lincoln, Neb.).

Ghizzetti, A. Sopra due particolari problemi misti di Dirichlet-Neumann per l'equazione di Laplace. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 1, 40–44 (1946).

Let T be the infinite angular opening $0<\theta<\omega<2\pi$, $0<r<\infty$. The mixed boundary value problem which the author solves is that of finding the harmonic function $u(r,\theta)$ in T such that $u(r,0)=f(r)$ and $u_\theta(r,\omega)=g(r)$, where f and g are given continuous functions, and such that $u\to 0$ when $r\to 0$ or $r\to\infty$ along rays, the approach being uniform when u is divided by $\log(\omega-\theta)$. The solution is given explicitly in terms of integrals of f and g and it is stated that it is unique. No proofs are given. The corresponding problem, in which the values of u and the normal derivative are given on complementary arcs of a circle, is also solved. This second problem reduces to the first by a conformal transformation. *J. W. Green.*

Friedrichs, K. O. An inequality for potential functions. Amer. J. Math. 68, 581–592 (1946).

Considérons dans l'espace euclidien $(x_1, \cdots, x_N)$ un domaine R dont la frontière est assez régulière mais peut présenter des arêtes et sommets. Soit u harmonique dans R satisfaisant à

$$\int_R\sum_i(\partial u/\partial x_i)^2dv<\infty, \qquad \int_R(\partial u/\partial x_1)dv=0$$

(dv élément de volume). Alors il existe une constante $\Gamma>0$ telle que

$$\int_R(\partial u/\partial x_1)^2dv\leqq\Gamma\int_R\{(\partial u/\partial x_2)^2+\cdots+(\partial u/\partial x_N)^2\}dv.$$

M. Brelot (Grenoble).

Rocco Boselli, Anna. Le funzioni di Green per gli iperstrati sferici. Rend. Sem. Mat. Univ. Padova 14, 5–16 (1943).

Let Γ_m be the region in m-dimensional space bounded by two concentric hyperspherical surfaces and let $g_m(M, M_0)$ be the Green's function for Γ_m with pole at M_0. The author finds, by the use of the iterated images of M_0 with respect to the two boundaries of Γ_m, explicit expressions for g_m in the form of infinite series. In the case $m=2$ it is shown that the series solution thus obtained reduces to the known formula for g_2 in terms of elliptic integrals. By examination of the terms of the series for g_m, an interesting recursion formula relating g_m and g_{m+2} is revealed. From considerations of symmetry, g_m is actually a function of three variables; namely, OM, OM_0, and the angle between OM and OM_0. Let x be the projection of OM on OM_0 and y the projection of OM on the hyperplane orthogonal to OM_0. Then g_m is also determined by x, y, and $OM_0=\rho$. It is proved that

$$g_{m+2}=\frac{1}{(m-2)\rho}\left(\frac{\partial g_m}{\partial x}-\frac{x}{y}\frac{\partial g_m}{\partial y}\right).$$

Thus it is possible to determine by differentiation all g_m from g_2 and g_3. *J. W. Green* (Los Angeles, Calif.).

Caligo, D. Sul problema di Dirichlet per l'iperstrato. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 1, 534–539 (1946).

Let $P=(X;y)=(x_1, x_2, \cdots, x_r; y)$ denote a point in the $(r+1)$-dimensional space $S_{r+1}=S_r\times S_1$ and T the slab of S_{r+1} determined by $0<y<\alpha$. Also let g_1 and g_2 be continuous functions of X and f a continuous function of P defined in the slab. Finally, let φ be a function of X; φ can be regarded as a function of P in which y does not appear. The author considers functions in T satisfying $\Delta u=f$, assuming g_1 and g_2 on the two boundaries of T, and such that $u(P)/\varphi(P)\to 0$ uniformly as $P\to\infty$, where the approach is via points $(X;y)$ with X lying on a ray in S_r. The object of the paper is to find what φ will assure that the function u, if it exists, is unique. It is found, by considering the expansion of u in hyperspherical functions, that $\varphi\sim e^{a\rho}/\rho^{\frac{1}{2}(r-1)}$ suffices, where $\rho^2=\sum_1^r x_i^2$. *J. W. Green* (Los Angeles, Calif.).

Cimmino, Gianfranco. Sul problema generalizzato di Dirichlet per l'equazione di Poisson. Rend. Sem. Mat. Univ. Padova 11, 28–89 (1940).

For a bounded open connected set A in n-dimensional space, $n\geqq 2$, whose boundary B consists of $m+1$ closed

connected sets, the author considers a generalized Dirichlet problem for the Poisson equation: given two functions φ and f, to determine a function u which satisfies $\Delta u = f$ in A and which assumes, in the mean, the values φ on B. Existence and uniqueness theorems are established under suitably restrictive hypotheses. *E. F. Beckenbach.*

Ridder, J. Über areolär-harmonische Funktionen. Acta Math. **78**, 205–289 (1946).

Considérons une fonction $u(x, y)$ admettant dans un domaine des dérivées premières (finies) continues, et, en un point P, le quotient λ par l'aire d'un carré de centre P (à cotés parallèles aux axes) du flux $\int(du/dn)ds$ entrant dans le carré. On introduit, lorsque le coté du carré tend vers 0, lim sup ou inf de λ et, s'il y a lieu, la limite finie dite dérivée aréolaire (dans quel cas u est dite aréolaire-harmonique en P). Cet opérateur [qu'il semblerait préférable de définir avec des cercles plutôt qu'avec des carrés], dont la nullité entraîne l'harmonicité de u et qui vaut $-\Delta u$ quand il y a des dérivées secondes (finies) continues, est étudié systématiquement et remplacera le Δu ordinaire dans diverses formules et problèmes classiques, par exemple dans le problème de Dirichlet pour $\Delta u = \varphi(x, y)$. Puis on introduit les dérivées aréolaires successives; la nullité de la p^e entraine que u soit polyharmonique d'ordre p; d'une manière générale ces dérivées remplaceront les Δ itérés ordinaires dans un prolongement de l'étude précédente étendant un peu la théorie des fonctions polyharmoniques. C'est seulement parmi quelques petites applications finales que l'on indique que l'opérateur étudié rentre dans la catégorie des vieux "laplaciens généralisés" (la condition du signe en un extremum n'était pas immédiate). Mais tous ces laplaciens ont peu d'intérêt maintenant que l'on sait comme cas particulier d'une théorie générale définir et utiliser le Δ d'une fonction localement sommable quelconque [voir L. Schwartz, Ann. Univ. Grenoble. Sect. Sci. Math. Phys. (N.S.) **21** (1945), 57–74 (1946); ces Rev. **8**, 264]. *M. Brelot.*

Cimmino, Gianfranco. Su alcuni sistemi lineari omogenei di equazioni alle derivate parziali del primo ordine. Rend. Sem. Mat. Univ. Padova **12**, 89–113 (1941).

Let $X(x, y)$ and $Y(x, y)$ be real functions satisfying

$$a_{11}X_x + a_{12}X_y + b_{11}Y_x + b_{12}Y_y = 0,$$
$$a_{21}X_x + a_{22}X_y + b_{21}Y_x + b_{22}Y_y = 0,$$

where the a's and b's are real constants and the subscripts indicate partial differentiation. If the a's and b's are such that all solutions X, Y are necessarily harmonic, then the equations can be solved: $X_x = b'_{11}Y_x + b'_{12}Y_y$, $X_y = b'_{21}Y_x + b'_{22}Y_y$, and in order that X, Y be harmonic whenever X, Y is a solution, it is necessary that $X - b'_{11}Y$ and $b'_{12}Y$ satisfy the Cauchy-Riemann differential equations.

In the case of three equations of the above type, in three unknown functions X, Y, Z with three independent variables x, y, z, the result obtained is that there are no systems all of whose solutions X, Y, Z consist of triples of harmonic functions.

In the case of four equations, the following system is considered, all solutions of which are easily shown to be harmonic: (1) $X_x - Y_y + Z_z - T_t = 0$, $X_y + Y_x - Z_t - T_z = 0$, $X_z - Y_t - Z_x + T_y = 0$, $X_t + Y_z + Z_y + T_x = 0$. The equations (2) $\int(Xdydzdt - Ydzdtdx + Zdtdxdy - Tdxdydz) = 0$ $\cdots$ (and three similar equations) are introduced. The integral is a triple integral over the boundary F of a region D. It is proved that, if X, Y, Z, T satisfy the four equations (2) for the boundary of every rectangular domain D for a neighborhood of each point of a region A, and in addition satisfy a Hölder condition at each point of A, then X, Y, Z, T are harmonic in A and satisfy (1). An analogue, for X, Y, Z, T, of the Cauchy integral formula for functions of a complex variable is next deduced. Next, a theorem on removable singularities is found. Then a similar treatment is given for the system $X_i^{(k)} = X_k^{(i)}$, $i, k = 1, \cdots, n$, $\sum_{k=1}^{n} X_k^{(k)} = 0$, where the subscript i indicates differentiation with respect to x_i. For any solution, every $X^{(k)}$ is harmonic. Finally, the case of m equations in n unknown functions and with n independent variables is treated, but without obtaining results analogous to those quoted above. *A. B. Brown.*

Obreschkoff, Nicola. Intorno alle funzioni armoniche sopra le superficie. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **4**, 420–423 (1943).

Obreschkoff, Nicola. Sulle funzioni armoniche sopra la sfera. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **4**, 133–136 (1943).

Obreschkoff, Nikola. Sulle funzioni armoniche sopra l'ipersfera. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **4**, 359–362 (1943).

A function φ defined on a surface is said to be harmonic provided $\Delta_2\varphi = 0$, where Δ_2 is the second Beltrami operator. These functions possess many properties closely analogous to those of ordinary harmonic functions. In the first paper, the author discusses some of these analogies and derives in a formal way appropriate forms of the Gauss and Green theorems, integral representation by Green's function and integral equations for the solution of Dirichlet problems. In the second paper these derivations are carried out in detail for the case of the sphere, and in the third paper, for the general hypersphere. *J. W. Green.*

Lelong, P. Les fonctions plurisousharmoniques. Ann. Sci. École Norm. Sup. (3) **62**, 301–338 (1945).

Dans un domaine D de l'espace numérique complexe à n dimensions C^n, une fonction f à valeurs numériques réelles $(-\infty \leqq f < \infty)$ est plurisousharmonique (en abrégé: psh) si: (a) f est semi-continue supérieurement; (b) sur toute "droite complexe," la restriction de f est sousharmonique (en abrégé: sh). On écarte le cas où f est la constante $-\infty$. Ces conditions ont un caractère local. L'enveloppe supérieure d'une famille finie de fonctions psh est psh; la limite d'une suite décroissante de psh est une psh ou la constante $-\infty$. La condition (b) peut être remplacée par: (b′) pour tout n-cercle intérieur à D et de centre $a = (a_i)$, $f(a)$ n'est pas supérieure à la valeur moyenne de f sur l'arête du n-cercle (un n-cercle est l'ensemble des $z = (z_i)$ tels que $|z_i - a_i| \leqq r_i$, ou son transformé par une transformation linéaire unitaire; l'arête est l'ensemble $|z_i - a_i| = r_i$ ou son transformé). Compte tenu de (b), on peut remplacer (a) par une condition en apparence plus faible: f est bornée supérieurement sur tout compact de D. En effet: si une $f(z_1, \cdots, z_n) \leqq 0$ est sh par rapport à chaque z_i séparément, f est semi-continue supérieurement [on peut simplifier la démonstration de l'auteur].

Pour que f deux fois continuement différentiable soit psh, il faut et il suffit que $\sum_{i,j}\alpha_i\bar{\alpha}_j\partial^2 f/\partial z_i\partial\bar{z}_j$ soit une fonction non négative quelles que soient les fonctions complexes α_i. Grâce à la théorie des "distributions" de L. Schwartz [Ann. Univ. Grenoble. Sect. Sci. Math. Phys. (N.S.) **21** (1945), 57–74 (1946); ces Rev. **8**, 264], que l'auteur semble ignorer, cette condition garde son sens même sans hypothèse de différentiabilité: elle exprime que $\sum_{i,j}\alpha_i\bar{\alpha}_j\partial^2 f/\partial z_i\partial\bar{z}_j$ est une "distribution" positive, donc définit des masses positives

dans l'espace. En particulier, $\sum_i \partial^2 f/\partial z_i \partial \bar{z}_i$ est un invariant vis-à-vis des transformations linéaires unitaires; il définit la distribution de masses dont le potentiel newtonien (dans l'espace des $2n$ variables réelles $x_i = \Re(z_i)$ et $y_i = \Im(z_i)$) est égal à f à l'addition près d'une fonction harmonique. La notion de fonction psh est invariante par toute transformation analytique complexe; elle vaut sur une variété (abstraite) à structure analytique complexe. En particulier, une psh de l'espace C^n induit une psh sur toute variété analytique située dans C^n.

Classes particulières de psh: (1) log $|g|$, où $g(z_1, \cdots, z_n)$ est analytique (holomorphe); la distribution de masses définie par $\sum_i \partial^2 f/\partial z_i \partial \bar{z}_i$ est portée par la variété $g=0$ et de densité superficielle constante [théorème de Poincaré, Acta Math. 22, 89–178 (1899); voir p. 159]; on retrouve en même temps le théorème de Wirtinger [Monatsh. Math. Phys. 44, 343–365 (1936)]: l'aire d'un morceau de cette variété est la somme des aires de ses projections sur les sous-espaces coordonnés de dimension (complexe) $n-1$; (2) les psh qui ne dépendent que des $|z_i|$ sont les

$$u(\log |z_1|, \cdots, \log |z_n|),$$

où u est une fonction convexe quelconque.

Les fonctions psh s'introduisent naturellement dans l'étude des singularités des fonctions analytiques. Par exemple, si D est un domaine d'holomorphie (domaine total d'existence d'une fonction holomorphe), la distance $\delta(z)$ d'un point $z \epsilon D$ à la frontière de D est telle que $-\log \delta(z)$ soit psh; la distance peut s'entendre au sens ordinaire (distance euclidienne), ou être comptée parallèlement à une "droite complexe" fixe. La fonction induite par $-\log \delta(z)$ sur toute variété analytique contenue dans D est aussi psh. Presque toutes les propriétés connues des domaines d'holomorphie résultent de celle-là. *H. Cartan* (Strasbourg).

Tolotti, C. Sulla struttura delle funzioni bi-iperarmoniche in tre variabili indipendenti. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **1**, 359–363 (1946).

Let u be a biharmonic function in a region D in three dimensions, that is, let u satisfy $\Delta^2 u = 0$ in D. The region D is said to be normal with respect to a plane π if every normal to π intersects D in an interval, which may be degenerate or vacuous; D is said to normal with respect to a pole Q if every ray issuing from Q intersects D in such an interval. Almansi [Ann. Mat. Pura Appl. (3) 2, 1–51 (1899)] showed that, if D is regular with respect to π, then u can be expressed in the form $u = u_0 + xu_1$, where u_i is harmonic and x is the coordinate orthogonal to π; also if D is regular with respect to Q, then u can be expressed as $u = u_0 + \rho^2 u_1$, where u_i is harmonic and ρ is the distance from Q. In the present paper the author shows by the construction of counter examples that regularity is necessary as well as sufficient to assure such representations for all functions biharmonic in D. It is also shown that, in any D, u can be written in the form $u = u_0 + \rho_1^2 u_1 + \rho_2^2 u_2 + \rho_3^2 u_3$, where u_i is harmonic and ρ_i denotes the distance from a point Q_i. *J. W. Green* (Los Angeles, Calif.).

Pastori, Maria. Sulle discontinuità dei potenziali poliarmonici. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **3**, 741–747 (1942).

Pastori, Maria. A proposito delle discontinuità dei potenziali poliarmonici. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) **8**(77), 165–172 (1944).

A polyharmonic potential of order m in three dimensions is computed just like the Newtonian potential, except that $1/r$ is replaced by r^{2m-3}. The result is a polyharmonic function u of order m, satisfying $\Delta^m u = 0$. In the present papers, the author studies the discontinuities of the derivatives of these potentials under the hypothesis that the densities involved are analytic. The results in the first paper are incorrect. In the second paper, this error is pointed out and the following correct results established. In the sequence of derivatives of u, discontinuities first appear in the $(2m-1)$th derivative in the case of a single layer, in the $(2m-2)$th in the case of a double layer, and in the $2m$th in the case of a volume distribution. *J. W. Green* (Los Angeles, Calif.).

Differential Equations

Friedrichs, K. O., and Wasow, W. R. Singular perturbations of non-linear oscillations. Duke Math. J. **13**, 367–381 (1946).

The system

$$x_i' = f_i(x_1, \cdots, x_n), \qquad i = 1, \cdots, n-1,$$
$$\epsilon x_n' = f_n(x_1, \cdots, x_n),$$

where the f_i have continuous first derivatives, is considered for small ϵ. Let the limiting system, i.e., with $\epsilon = 0$, have an "isolated" periodic solution $x_i = u_i(t)$ and let

$$\frac{\partial f_n(u_1(t), \cdots, u_n(t))}{\partial x_n} \neq 0.$$

Then for small ϵ the original system admits a closed continuous integral curve $x_i = v_i(t; \epsilon)$ such that (a) as $\epsilon \to 0$, $v_i \to u_i$, (b) $x_i = v_i(t; \epsilon)$ possess a period which tends to the period of u_i as $\epsilon \to 0$, (c) the $v_i(t; \epsilon)$ are continuous functions of ϵ for small ϵ. *N. Levinson* (Cambridge, Mass.).

Butlewski, Z. Sur les intégrales bornées des équations différentielles. Ann. Soc. Polon. Math. **18**, 47–54 (1945).

In the differential equation

$$\frac{d}{dt}\left\{\theta(t)\frac{dx}{dt}\right\} + \sum_{i=0}^{m} A_{2i+1}(t) x^{2i+1} = 0,$$

$\theta(t)$ and A_{2i+1} are positive continuous differentiable functions for $t \geqq t_0 > 0$. The author shows that, if $A_{2i+1}(t)\theta(t) > 0$ for $i = 0, 1, \cdots, m$ and for $t \geqq t_0$, then $x(t)$ is bounded as $t \to +\infty$. The case where the equation begins with the terms $x'' + B(t)x'$ is also considered. Finally a system of order n is considered and an analogous result obtained.

N. Levinson (Cambridge, Mass.).

Wintner, Aurel. Asymptotic integration constants. Amer. J. Math. **68**, 553–559 (1946).

Vectors f and x, each having a finite number k of real components, are related by $f = f(x, t)$, where $f(x, t)$ is defined and continuous at every point (x, t) of the whole Euclidean x-space and the half-line $0 \leqq t < \infty$. Let functions $\lambda(r)$ and $\phi(r)$ exist on $0 \leqq t < \infty$ and be continuous and positive on $0 < t < \infty$. Let $\int_0^\infty \lambda(r) dr < \infty$, $\int_1^\infty \{\phi(r)\}^{-1} dr = \infty$, $|f(x, t)| \leqq \lambda(t)\phi(|x|)$, where absolute value signs refer to Euclidean length. Under the conditions stated, the following results are established. (i) Every solution $x \equiv x(t)$ of the differential system $x' = f(x, t)$, $x(0) = x^0$, where x^0 is arbitrarily given, exists for $0 < t < \infty$ and tends to a finite limit vector $x(\infty)$ as $t \to \infty$. (ii) For ${}^0x$ arbitrarily given, the differential system $x' = f(x, t)$, $x(\infty) = {}^0x$, has a solution $x = x(t)$, $0 \leqq t < \infty$. Dual conditions and resulting theorems are considered in an appendix. *W. M. Whyburn.*

Haag, Jules. Sur la stabilité des solutions de certains systèmes d'équations différentielles. Bull. Sci. Math. (2) 70, 21–36 (1946).

The author considers first a canonical system of differential equations $udy_j/du=f_j(u, y_1, \cdots, y_m)$, where $j=1, \cdots, m$; the independent variable u is real; the dependent variables $y_1, \cdots, y_m$ are complex; the functions $f_j(u, y_1, \cdots, y_m)$ are defined for $0\leqq u\leqq u_0$ and $|y_j|\leqq b$ and are of the form $S_j(y_j+y_{j-1})+g_j(u, y_1, \cdots, y_m)$ or $S_jy_j+g_j(u, y_1, \cdots, y_m)$; the S_j are constants and the functions $g_j(u, y_1, \cdots, y_m)$ possess continuous first derivatives vanishing at $u=y_1=\cdots=y_m=0$. The proofs use only the estimates implied by the existence and continuity of these partial derivatives, rather than the analyticity of g_j with respect to y_k. It is shown that, if s_j, the real part of S_j, is different from zero for all j and if u_0 and b are sufficiently small, then, corresponding to every choice of the initial constants $y_k(u_0)$ belonging to those indices k for which $s_k>0$, there is a unique solution of the system such that $y_j(u)\to 0$ as $u\to 0$ for all j. (The proof involves the method of successive approximations.) In particular, if all $s_j>0$, the general solution satisfies $y_j(u)\to 0$ as $u\to 0$ for all j. Also, when $s_j>0$ for all j, then, for any two solutions $y_j(u)$ and $y_j^*(u)$, one has $|y_j(u)-y_j^*(u)|=O(u^r)$ as $u\to 0$ whenever $r<\min(s_1, \cdots, s_m)$. [For sharper results in the case $m=1$, cf. Hartman and Wintner, Amer. J. Math. 68, 301–308 (1946); Hartman, ibid., 495–504 (1946); these Rev. 7, 444; 8, 71]. The author uses his general result to deduce a refinement of a theorem of Liapounoff dealing with stability of solutions of the system $dx_j/dt=\sum_{k=1}^{m}a_{jk}x_k+\phi_j(t, x_1, \cdots, x_m)$ as $t\to\infty$.

He also considers some variations on his main result; for example, the case where one of the constants $S_j=0$; or where some S_j are purely imaginary, although no asymptotic behavior of the solutions is proved in this case; or finally, where udy_j/du on the left side of the differential equations are replaced by $f(u)dy_j/du$, the function $f(u)$ being continuous and satisfying $f(u)=O(u^r)$ for some $r>1$.

P. Hartman (Baltimore, Md.).

Scorza Dragoni, Giuseppe. Un'osservazione su un problema per le equazioni differenziali ordinarie. Ist. Veneto Sci. Lett. Arti. Parte II. Cl. Sci. Mat. Nat. 101, 203–212 (1942).

This is closely related to a previous note by Zwirner [Rend. Sem. Mat. Univ. Padova 12, 114–122 (1941); these Rev. 8, 206]. Using essentially the same method as Zwirner, the author proves a further theorem concerning the existence of a solution of the problem

$$y^{(n)}=f(x, y, y', \cdots, y^{(n-1)}),$$
$$y(x_1)=c_1,\ y(x_2)=c_2,\ \cdots,\ y(x_n)=c_n.$$

L. A. MacColl (New York, N. Y.).

Zwirner, Giuseppe. Problemi di valori al contorno per sistemi di equazioni differenziali ordinarie. Ist. Veneto Sci. Lett. Arti. Parte II. Cl. Sci. Mat. Nat. 101, 405–418 (1942).

In a previous note [Rend. Sem. Mat. Univ. Padova 12, 114–122 (1941); these Rev. 8, 206] the author has employed a novel method to establish the existence of solutions of problems of the form described in the preceding review. In the present note he employs the method to prove the existence of solutions of problems of the same kind which involve more than one dependent variable.

L. A. MacColl (New York, N. Y.).

Zanaboni, O. Soluzioni particolari delle equazioni differenziali lineari non omogenee, a coefficienti costanti. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) 6(75), 540–546 (1942).

Nadolschi, Victor L. Sur une nouvelle classe d'équations différentielles. Ann. Sci. Univ. Jassy. Sect. I. 27, 289–302 (1941).

An investigation of certain second order differential equations which can be reduced to linear differential equations with constant coefficients by a transformation of the independent variable. *P. Franklin* (Cambridge, Mass.).

Casale, Ambrogio. Risoluzione di un'equazione differenziale a derivate ordinarie del prim'ordine. Boll. Un. Mat. Ital. (2) 5, 235–236 (1943).

The equation is $y+\varphi(y')+xy'^2\psi(xy'-y)=0$.

Belardinelli, G. Su una equazione differenziale. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) 4(73), 318–326 (1940).

The author considers the problem of representing solutions of $t(a_1+b_2t+c_2t^2)y''+(a_1+b_1t+c_1t^2)y'+(h+h't)y=0$ by means of functionals of the form

$$f(x)=(2\pi i)^{-1}\int_C t^{-1}F(x/t)\phi(t)dt,$$

where $F(z)=\sum_{n=0}^{\infty}R(n)z^n$, and $R(n)$ is a rational function of n. *R. Bellman* (Princeton, N. J.).

Bristow, Leonard. Expansion of functions in combinations of generalized hypergeometric functions. Duke Math. J. 13, 331–344 (1946).

The paper is concerned with the boundary problem associated with a differential equation

$$y^{(n)}+*+a_2x^{-2}y^{(n-2)}+\cdots+a_nx^{-n}y=0$$

and linear boundary conditions applying at points $x=\alpha$ and $x=\beta$, with $\beta>0$ and $\alpha<0$ or $\alpha=0$. The coefficients a_j are restricted to be constants, and moreover such that the equation is self-adjoint, and has, relative to $x=0$, indices with real parts between 0 and $n-1$. The principal objective is that of showing that this problem, and the corresponding one based upon the equation $y^{(n)}+\lambda y=0$ with regular boundary conditions, yield expansions for arbitrary functions that are equiconvergent.

There are some misprints in lemma 1 of the paper. Even aside from these, the reviewer seems unable to dispel concern upon the applications of the lemma, or indeed, upon whether the Stokes phenomenon, which is operative in this problem, has been adequately regarded. *R. E. Langer*.

Merli, Luigi. Un criterio sufficiente di esistenza e di unicità per una classe di problemi ai limiti relativi alle equazioni differenziali lineari omogenee della forma:

$$\frac{d^m}{dx^m}[A_0y^{(m)}]+\frac{d^{m-1}}{dx^{m-1}}[A_1y^{(m-1)}]+\cdots+\frac{d}{dx}[A_my']+A_my=0.$$

Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) 7(76), 261–270 (1943).

Luzin, N. N. A study of the matrix theory of differential equations. Avtomatika i Telemehanika 1940, 4–66 (1940). (Russian) no 5

An elementary and well written treatment, for technicians and engineers, of systems $\sum a_{ij}(D)x_j=f_i(t)$, $D=d/dt$, where

the a_{ij} are polynomials in D. The treatment is by the general method of λ-matrices. *S. Lefschetz.*

Baiada, E. Alcune considerazioni sull'esistenza della soluzione delle equazioni alle derivate parziali. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 1, 296–301 (1946).

The author shows that the Cauchy problem

$$\partial z/\partial x = f(x, y, z, \partial z/\partial y), \quad z(0, y) = 0,$$

may not have a solution with $\partial z/\partial x$, $\partial z/\partial y$ bounded, if the only assumption on $f(x, y, z, p)$ is that it is a continuous function of x, y, z, p. *F. G. Dressel* (Durham, N. C.).

Faedo, S. Sui problemi d'equilibrio della fisica-matematica. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 1, 343–345 (1946).

Announcement of a forthcoming paper on the solution of boundary value problems by the Ritz method.

Bergman, Stefan. Classes of solutions of linear partial differential equations in three variables. Duke Math. J. 13, 419–458 (1946).

L'auteur reprend de manière indépendante ses travaux dérivant de l'idée suivante: chercher et étudier des opérateurs qui transforment les fonctions f d'un certain type en solutions Φ d'une certaine équation aux dérivées partielles dont on pourra par cette voie approfondir les propriétés. [Renvoyons surtout pour le cas de deux variables à Trans. Amer. Math. Soc. 57, 299–331 (1945); ces Rev. 7, 16; et pour le cas elliptique à 3 variables à Trans. Amer. Math. Soc. 59, 216–247 (1946); ces Rev. 7, 448.] Ici l'auteur traite le cas plus général de l'équation à trois variables:

$$\Delta\Phi + A(r^2)\sum x_i \partial\Phi/\partial x_i + c(r^2)\Phi = 0.$$

L'opérateur qui généralise celui de Whittaker est analogue à celui du cas de deux variables qui inspire le présent développement où l'on cherche encore à "transporter" des propriétés des f aux Φ. *M. Brelot* (Grenoble).

Cinquini-Cibrario, Maria. Equazioni ellittico-paraboliche in dominii infiniti. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) 6(75), 619–629 (1942).

Using results from several earlier papers the author writes theorems relative to the solutions of each of the equations

$$(1) \qquad k^2 y^{2(k+1)} u_{xx}(x, y) + u_{yy} = 0,$$

$$(2) \qquad k^{-2} x^{2(k+1)} u_{xx} + u_{yy} + 2k^{-2} x^{2k+1} u_x = 0,$$

$$(3) \qquad k^{-2} x^{2(k+1)} u_{xx} + u_{yy} = 0, \qquad k = 2, 3, \cdots.$$

A typical example of the theorems is the following. Let γ be a continuous curve without double points lying in the half plane $y>0$ and having $x=a$ and $x=b$ as asymptotes and let Δ be the region bounded by γ and the line $y=+\infty$. Then there exists a unique solution of (1) in Δ taking on preassigned continuous values on the curve γ. Such a solution of (1) will reduce to a linear function of x on the segment $a \leqq x \leqq b$ of the line $y=+\infty$. *F. G. Dressel.*

Sestini, Giorgio. Sopra un problema di propagazione del calore. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) 6(75), 47–65 (1942).

Let R_1, R_2, R_3 be the surfaces of three coaxial cylinders of radii $r_1<r_2<r_3$, respectively, and S_1 the region interior to R_1, while S_2 and S_3 are the regions between R_1, R_2 and R_2, R_3, respectively. The regions S_1, S_2, S_3 contain different materials and region S_2 is supplied continuously with $K(t)$ units of heat. Some well-known boundary value problems for the heat equation are worked out both for the stationary ($\partial u/\partial t=0$) and the nonstationary state. If $u_i(r, t)$ represents the temperature function in region S_i, the author then finds the limiting functions u_1^*, u_3^* of u_1 and u_3, when the region S_2 shrinks to zero and the supply of heat $K(t)$ to S_2 becomes infinite in such a manner that $\lim_{r_2\to r_1}(r_2-r_1)K(t)$ remains finite. *F. G. Dressel* (Durham, N. C.).

Parodi, Hippolyte. Nouvelle solution du problème du mur plan indéfini, soumis, sur ses deux faces, à des variations périodiques de température. C. R. Acad. Sci. Paris 223, 472–474 (1946).

In this note the author proposes to derive a new solution of the simple heat equation of the form $\theta(x, t) = A(x)\sin\omega t + B(x)\cos\omega t$, where ω is a prescribed constant. Although the problem of determining all functions $A(x)$ and $B(x)$ in this case is an elementary, short and direct one with a well-known result, a comparatively long and formal method is presented here. Owing to an error in sign the result is incorrect. In the author's expressions for $A(x)$ and $B(x)$ the coefficient m, described as a real constant, must be imaginary if his function is to satisfy the heat equation.
R. V. Churchill (Ann Arbor, Mich.).

Parodi, Hippolyte. Sur le problème du refroidissement de la sphère. C. R. Acad. Sci. Paris 223, 540–542 (1946).

A variation of the usual series method of deriving formulas for the temperatures $\theta(r, t)$ in a hollow sphere with prescribed boundary and initial conditions is outlined. The method is a formal one which introduces differential recurrence relations. The details are not presented.
R. V. Churchill (Ann Arbor, Mich.).

Faggiani, Dalberto. Oscillazioni di temperatura nei tubi termicamente isolati. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) 5(74), 491–500 (1941).

Faggiani, Dalberto. Trasmissione di calore in regime permanente e periodico nei tubi alettati. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) 5(74), 389–402 (1941).

Amerio, L. Sull'equazione del calore. I. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 1, 346–352 (1946).

Amerio, L. Sull'equazione del calore. II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 1, 544–548 (1946).

For the parabolic equation

$$(1) \quad E(u) \equiv \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} - \frac{\partial u}{\partial y} = f(x, y), \qquad (x, y) = (x_1, \cdots, x_n, y),$$

let τ be a region with boundary σ for which the formula of Green applies to (1). Let n and v be, respectively, the normal and conormal of σ oriented interior to τ, and write the integral

$$H(A, B; F(\xi, \eta)) = \int_\sigma \left\{ A(\xi, \eta)\frac{\partial F(\xi, \eta)}{\partial v}\sin(n, \eta) + A(\xi, \eta)F(\xi, \eta)\cos(n, \eta) - B(\xi, \eta)F(\xi, \eta)\sin(n, \eta)\right\} d\sigma - \int_\tau f(\xi, \eta)F(\xi, \eta)d\tau.$$

It follows from the formula of Green that, if $G(\xi, \eta; x, y)$ is

the fundamental solution of (1) with normalizing factor $1/(2\pi^{\frac{1}{2}})^m$ and $u(x, y)$ is a solution of (1), then

(2) $H(A, B; G(\xi, \eta; x, y))=0, \qquad (x, y)$ exterior to τ,

(3) $\qquad\qquad = u(x, y), \qquad (x, y)$ interior to τ,

if $A=u$, $B=\partial u/\partial v$ on σ. The paper states conversely that, if A, B satisfy certain continuity conditions and are such that (2) holds, then (3) gives a solution of (1) interior to τ with $A=u$ and $B=\partial u/\partial v$ almost everywhere on σ.

Let $w_r = x_1^{\alpha_1}\cdots x_n^{\alpha_n}y^{\alpha_{n+1}}$ ($\alpha_k=0, 1, \cdots$) and let $E^*(u)$ be the adjoint of $E(u)$. If for all w_r the functions A, B, and u have certain continuity properties and are such that $-\int_\tau uE^*(w_r)d\tau=H(A, B, w_r)$, then u is a solution of (1) in τ.

Let σ_1 be that part of σ for which $\cos(n, y)=1$, σ_2 that part of σ for which $\sin(n, y)\neq 0$, and $h(x, y)$, $k(x, y)$, $C(x, y)$, $R(x, y)$ given functions ($h^2+k^2>0$). The problem of finding a solution $u(x, y)$ of (1) such that $u=R$ on σ_1, $hu+k\partial u/\partial v=C$ on σ_2, is solved by use of an infinite series of integral equations of the Fischer-Riesz type [M. Picone, Ann. Sci. Univ. Jassy **26**, 183–232 (1940); these Rev. **1**, 236]. Conditions for the uniqueness of this problem are also stated. Proofs will appear in another paper. *F. G. Dressel.*

Conforto, Fabio. Sulle deformazioni elastiche di un diedro omogeneo e isotropo. Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. (2) **70**, 163–233 (1942).

M. Picone has given a method for the solution of a great variety of physical problems [see, for example, Ann. Sci. Univ. Jassy **26**, 183–232 (1940); these Rev. **1**, 236]. By this method, the method of moments or integral method, a system of linear partial differential equations with various types of boundary conditions is transformed into an infinite system of integral equations of the Fischer-Riesz type. Moreover, Picone shows how to solve this system of integral equations or, at least, how to obtain an approximation to the solution. The method does not in general give a proof for the existence of the solution, but it yields a series converging to the desired solution in all cases where it exists, and it makes it possible to consider very complicated problems. Such a problem is the determination of the elastic deformations of a homogeneous isotropic dihedral by a force perpendicular to one of the faces, applied at a point O of the edge. The author gives the preliminary results of an investigation of this problem, with an exposition of the method as it works in this particular case.

The equations of elasticity are

$$(\lambda+\mu)\left(\frac{\partial^2 u_1}{\partial x'^2}+\frac{\partial^2 u_2}{\partial x'\partial y'}+\frac{\partial^2 u_3}{\partial x'\partial z'}\right)+\mu\left(\frac{\partial^2 u_1}{\partial x'^2}+\frac{\partial^2 u_1}{\partial y'^2}+\frac{\partial^2 u_1}{\partial z'^2}\right)=0,$$

$$(\lambda+\mu)\left(\frac{\partial^2 u_1}{\partial x'\partial y'}+\frac{\partial^2 u_2}{\partial y'^2}+\frac{\partial^2 u_3}{\partial y'\partial z'}\right)+\mu\left(\frac{\partial^2 u_2}{\partial x'^2}+\frac{\partial^2 u_2}{\partial y'^2}+\frac{\partial^2 u_2}{\partial z'^2}\right)=0,$$

$$(\lambda+\mu)\left(\frac{\partial^2 u_1}{\partial x'\partial z'}+\frac{\partial^2 u_2}{\partial y'\partial z'}+\frac{\partial^2 u_3}{\partial z'^2}\right)+\mu\left(\frac{\partial^2 u_3}{\partial x'^2}+\frac{\partial^2 u_3}{\partial y'^2}+\frac{\partial^2 u_3}{\partial z'^2}\right)=0,$$

where λ and μ are constants depending on the material. The edge of the dihedral is chosen as the axis of z'; O is the origin of coordinates and the x'-axis has the direction of the applied force. The boundary conditions are given by rather complicated equations. Introducing the operators

$$D=\partial^2/\partial z^2-(1+z^2), \quad \Delta_x=\partial^2/\partial x^2-1, \quad \Delta_y=\partial^2/\partial y^2-1,$$

the author first gives the inverses of these by integral operations, where, of course, Green's functions occur under the integral sign. The passage to the integral equations is prepared by introducing a new set of unknowns

$$U_i(x, y, z)=D\Delta_x\Delta_y u_i(x, y, z), \quad \alpha_i(y, z)=D\Delta_y u_i(0, y, z),$$
$$\beta_i(x, z)=D\Delta_x u_i(x, 0, z), \quad \omega_i(z)=Du_i(0, 0, z).$$

The computation of the various coefficients in the integral equations and the subsequent numerical calculations require a considerable amount of work, the results of which are given here to the first approximation. *H. Bremekamp.*

Aquaro, G. Su un problema riducibile di integrazione di una particolare equazione a derivate parziali. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **1**, 530–534 (1946).

Let R be the region ($x_1\leqq x\leqq x_2$; $y_1\leqq y\leqq y_2$) and let Γ be the class of solutions $u(x, y)$ satisfying

(1) $u_{xxy}=u_y+a(x, y)u_{xx}+b(x, y)u_x+c(x, y)u+f(x, y)$

such that u, u_x, u_{xx}, u_y, u_{xxy} are continuous in R (a, b, c, f are given continuous functions of x and y). The following boundary value problem is reduced to solving an integral equation. Find, in the class Γ, a solution of (1) interior to R such that

$$u(x, y_1)=X(x), \qquad x_1\leqq x\leqq x_2,$$

(2) $u_x(x_i, y)=Y_i(y), \qquad y_1\leqq y\leqq y_2, \quad i=1, 2,$

$$X'(x_1)=Y_1(y_1), \quad X'(x_2)=Y_2(y_1),$$

where X, Y_1, Y_2 are given continuous functions and X has a continuous second derivative. The author notes that the boundary conditions (2) would also determine the solution of the second order parabolic differential equation.

F. G. Dressel (Durham, N. C.).

Mangeron, D. New method for determining the integrals of linear partial differential equations by application of the multiple Laplace transformation with finite domain of integration. Revista Ştiinţifică "V. Adamachi" **32**, 38–40 (1946). (Romanian. English summary)

Mangeron, D. Green's function of order p in the theory of total differential equations of higher order. Revista Ştiinţifică "V. Adamachi" **32**, 40–42 (1946). (Romanian. French summary)

The word "total" appears to be a misprint for "partial."

Functional Analysis, Ergodic Theory

Gomes, A. Pereira. Introduction to the notion of functional in spaces without points. Portugaliae Math. **5**, 1–120 (1946). (Portuguese)

The usual definition of a functional $f(x)$ over a space E associates with every point $x\epsilon E$ a real number $f(x)$. Here the subsets of E form an atomic Boolean algebra. The present work considers the problem of defining functionals over Boolean algebras which are not atomic or, expressed in other terms, over spaces without points. The classical and most important instance of such a functional is the Lebesgue integral of a fixed function over a variable measurable set. In this situation the measurable sets modulo the sets of measure zero constitute a σ-Boolean algebra without points. The principal previous contributions in this field are due to C. Carathéodory [S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. **1938**, 27–69] and J. M. H. Olmsted [Trans. Amer. Math. Soc. **51**, 164–193 (1942); these Rev. **4**, 11].

In order to extend the notion of functional, it is first of all transformed as follows. In the classical case we may con-

sider the set $M(\lambda)$ of all x such that $f(x)\geqq\lambda$ and the set $N(\lambda)$ of all x such that $f(x)>\lambda$. These sets, for $-\infty\leqq\lambda\leqq\infty$, yield the reconstruction of the original $f(x)$. As a matter of fact, it is sufficient to know these sets only for a denumerably dense set of real numbers. This reformulation gives the starting point of the present investigation.

A functional is defined as above by means of the sets $M(\lambda)$ or $N(\lambda)$ (subject to certain monotonicity restrictions) on a σ-Boolean algebra E. For two such functionals f_1 and f_2 we write $f_1\leqq f_2$ in case $M_1(\lambda)\subset M_2(\lambda)$ for all λ. The totality of functionals φ constitutes a σ-structure (lattice) which is without atoms and not complemented. Algebraic operations may be introduced into φ in the following way. To define $f=f_1+f_2$ consider the set

$$N(\lambda)=\bigcup_{\lambda_n} N_1(\lambda_n)\cap N_2(\lambda-\lambda_n),$$

where the λ_n are a denumerable dense set of numbers. This definition is satisfactory in case f_1 and f_2 do not assume infinite values. As defined, addition has the desired properties. With the help of the Jordan decomposition, $f=f^+-f^-$, where $f^+=f\vee 0$ and $f^-=-f\vee 0$, one may proceed to define multiplication. First the operation is defined for positive functionals ($f=f^+$); then for arbitrary functionals.

Now suppose the basic σ-Boolean algebra E possesses a closure topology. This will be indicated by $[E, (^-)]$, where the closure operator satisfies (0) $\bar{0}=0$, (1) $\bar{A}\supset A$, (2) $\overline{A\cup B}=\bar{A}\cup\bar{B}$, (3) $\bar{\bar{A}}=\bar{A}$. We may now define the upper limit $\bar{f}$ of a functional f by means of $M_{\bar{f}}(\lambda)=\bigcap_{\lambda_n<\lambda}\overline{M_f(\lambda_n)}$. The inferior limit $\underline{f}$ is defined similarly. We have $\underline{f}=-\overline{(-f)}$; $\underline{f}\leqq\bar{f}$; $\overline{f\vee g}=\bar{f}\vee\bar{g}$; $\bar{\bar{f}}=\bar{f}$. Thus the operation of taking upper limits is a closure operation in φ. If $\bar{f}=f$, then f is said to be upper semi-continuous. The definitions of lower semi-continuity and continuity are obvious. Note that (naturally) these are "global" and not local descriptions of f. It is proved that a σ-Boolean algebra has the Cantor property (that every monotone descending chain of nonempty closed sets has a nonempty intersection) if and only if every upper semi-continuous functional attains its upper bound. In order that every continuous functional should have the Darboux property (of assuming all values between any two), it is necessary and sufficient that the basic topological algebra $[E, (^-)]$ should be connected. Finally it is shown that the Baire functionals on $[E, (^-)]$ are precisely the Borel sets of the structure $[\varphi, (^-)]$. *E. R. Lorch.*

Gomes, A. Pereira. Correction to the article "Introduction to the study of the notion of functional in spaces without points." Portugaliae Math. 5, 218 (1946). (Portuguese)

In the paper reviewed above the author proposed a problem which he stated incorrectly. Here he gives an amended version of the problem. *E. R. Lorch.*

Fan, Ky. Le prolongement des fonctionnelles continues sur un espace semi-ordonné. Revue Sci. (Rev. Rose Illus.) 82, 131–139 (1944).

Using terms of G. Birkhoff's "Lattice Theory" [Amer. Math. Soc. Colloquium Publ., v. 25, New York, 1940; these Rev. 1, 325], let S be a lattice, H and K sublattices of S, L the point set sum of H and K, and define a neighborhood in S to consist of all $s\epsilon S$ such that $h\leqq s\leqq k$ for fixed $h\epsilon H$, $k\epsilon K$; also suppose every s lies in some such neighborhood. Then S is a Hausdorff space, L is dense in S, and hence, by Fréchet's procedure [Fund. Math. 7, 210–224 (1925)], for each real function ϕ monotone increasing and continuous on L there is a maximal domain $P\subset S$ on which a continuous extension Φ of ϕ exists. The author shows that (1) P is a sublattice and Φ is monotone increasing; (2) if, also, ϕ on H and ϕ on K are modular then Φ is modular. If S is σ-complete and ϕ is as in (2) he furthermore proves that (3) if H is closed under countable meets and $\{h_n\}\downarrow$ implies $\phi(h_n)\downarrow\phi(\lim h_n)$, then P is closed under countable meets and $\{p_n\}\downarrow$ implies $\Phi(p_n)\downarrow\Phi(\lim p_n)$; (4) the dual of (3) holds with H replaced by K; (5) if S is a σ-complete Boolean algebra then P is also. Concluding examples show how this method generates the conventional limit for real sequences, Lebesgue measure and integration, and means for random variables. The paper's contents are generally related to the work of G. Birkhoff, von Neumann, Kantorovitch, Wilcox, Smiley and others on metric and lattice completions of lattices [cf. "Lattice Theory"] and to Daniell's approach to integration [Ann. of Math. (2) 19, 279–294 (1918); 21, 203–220 (1920)]. *B. J. Pettis.*

***v. Sz. Nagy, Béla. Spektraldarstellung linearer Transformationen des Hilbertschen Raumes.** Ergebnisse der Mathematik und ihrer Grenzgebiete, 5, no. 5. Springer, Berlin, 1942. iv+80 pp.

The purpose of this book is to derive the spectral representation theorem for normal operators in Hilbert space. Most of the methods and results have appeared in print before (though some only recently); they are collected here and presented in a very polished form. All the simplifications and rearrangements in proofs, discovered since von Neumann's pioneer work in the field (mostly by F. Riesz and the author), have been incorporated in the exposition. The style of the booklet makes for easy reading even though it is concise almost to the extent of being telegraphic. (It takes the author one page to define Hilbert space, orthogonality and distance and to prove the Schwarz inequality, the triangle inequality and the continuity of the inner product.)

The author makes good use of Riesz's useful comment that in a convex set a minimizing sequence with respect to the function $\|f\|$ is convergent. The positive square root of a bounded positive transformation A is defined by the iterative scheme $B_{n+1}=B_n+\frac{1}{2}(A-B_n^2)$. After a careful discussion of projections, spectral families and integrals of bounded Baire functions with respect to spectral families, the spectral representation of a bounded self-adjoint transformation is obtained very quickly. The positive part of A is defined as $A^+=\frac{1}{2}(+\sqrt{A^2}+A)$, and $E(\lambda)$ as the projection on the null space of $(A-\lambda I)^+$. Two proofs are given for the unbounded case: one parallels that due to F. Riesz and Lorch and the other is von Neumann's. The later chapters contain discussions of the proper value theory and completely continuous transformations (but not the general theory of multiplicity of continuous spectra), the functional calculus, Stone's theorem on the representation of unitary groups, and some of its extensions to semigroups of self-adjoint and normal transformations. The booklet concludes with a three page bibliography which lists most of the important modern contributions to the field.

P. R. Halmos (Chicago, Ill.).

Hamburger, Hans Ludwig. On a class of Hermitian transformations containing self-adjoint differential operators. Ann. of Math. (2) 47, 667–687 (1946).

The paper is an elaboration of an earlier abstract [Proc. Nat. Acad. Sci. U. S. A. 31, 185–189 (1945); these Rev. 7,

126; we refer to this review for all terms not explained here]. The first chapter is devoted to the proof of the following main theorem. Let H be an mth order transformation of class P in $\mathfrak{L}_2(\Delta)$ for which (a) the eigen functions $\varphi_s(x)$ have absolutely continuous derivatives up to and including the $(m-1)$th, while the mth derivatives exist almost everywhere and belong to $\mathfrak{L}_2(\Delta)$, and the $\psi_s(x)$ are continuous in Δ; (b) for any pair of interior points s, t of Δ the intersection $\mathfrak{D}_s \cap \mathfrak{D}_t$ is everywhere dense in $\mathfrak{D}$. Then H can be represented by a Sturm-Liouville operator of mth order, i.e., a formally self-adjoint differential operator (without the boundary conditions which would make it a self-adjoint operator in the sense of abstract Hilbert space).

The second chapter deals with two extensions of the concept of an mth order operator of class P. The first one is aimed at the case with singular endpoints and correspondingly an "mth order transformation of the generalized class P" is essentially defined to be one for which for each interval Δ' given by $a' \leqq x \leqq b'$ (a', b' interior points of Δ) the transformation $P_{\Delta'}HP_{\Delta'}$ is an mth order transformation of class P with respect to the interval Δ' if $P_{\Delta'}=P_{b'}-P_{a'}$. The second extension deals with any abstract Hilbert space $\mathfrak{H}$ instead the space $\mathfrak{L}_2(\Delta)$. Let $P(s)$ be a resolution of the identity defined in $\mathfrak{H}$ for every s of the interval $\Delta(a \leqq s \leqq b)$ such that the integral with respect to $dP(s)$ over Δ is the identity and the integral with respect to $sdP(s)$ over Δ is a self-adjoint operator K with the following properties: (i) K has no eigenvalues, (ii) K has a simple spectrum, (iii) no point of Δ belongs to the resolvent set of K. A closed Hermitian operator H in $\mathfrak{H}$ is then called an mth order transformation of class P if there exists a resolution of the identity $P(s)$ with the properties just described and with respect to which H satisfies the conditions set forth in the definition of an mth order transformation of class P [see the earlier review] after, in this definition, P_s, P_t, $\mathfrak{L}_2(\Delta)$ have been replaced by $P(s)$, $P(t)$, $\mathfrak{H}$, respectively. For both extensions theorems analogous to the main theorem are proved. *E. H. Rothe* (Ann Arbor, Mich.).

Krein, M. Concerning the resolvents of an Hermitian operator with the deficiency-index (m, m). C. R. (Doklady) Acad. Sci. URSS (N.S.) **52**, 651–654 (1946).

The author outlines a method of obtaining the forms of the generalized resolvents R_z of a Hermitian operator A on a Hilbert space $\mathfrak{H}$ with deficiency index (m, m), generalizing the result for deficiency index $(1, 1)$ obtained in a previous paper [same C. R. (N.S.) **43**, 323–326 (1944); these Rev. **6**, 131]. Generalized resolvents R_z correspond to spectral functions $E(t)$ of A by the formula $R_z\varphi = \int_{-\infty}^{\infty}(t-z)^{-1}dE(t)\varphi$ for $\Im z>0$. The set of all generalized resolvents of A is given by

$$R_z = R_z^0 + \sum_{j,k=1}^{m} [\cdot, \varphi_j(\bar{z})]h_{jk}(z)\varphi_k(z), \qquad \Im z>0,$$

where R_z^0 is the resolvent of a certain self-adjoint extension A^0 of A, $\varphi_j(z)$ are m linearly independent solutions of $A^*\varphi - z\varphi = 0$ (in $\mathfrak{H}$ for each nonreal z), where A^* is the operator maximally adjoined to A; $h_{jk}(z)$ forms a matrix of complex-valued functions of z, the reciprocal of the matrix of $q_{jk}(z)+f_{jk}(z)$, with

$$q_{jk}(z) = [(z-z_0)\varphi_j(z)+iy_0\varphi_j(\bar{z}_0),\ \varphi_k(z_0)],$$

z_0 is an arbitrary regular point of R_z^0, and $f_{jk}(z)$ are holomorphic and such that $\Im(\sum_{jk}\xi_j f_{jk}(z)\bar{\xi}_k) \geqq 0$ for $\Im z>0$. The case in which the operator A is positive, i.e., $[A\varphi, \varphi] \geqq 0$ for φ in $\mathfrak{D}(A)$, is also discussed. *T. H. Hildebrandt.*

Krein, M. On a general method of decomposing Hermite-positive nuclei into elementary products. C. R. (Doklady) Acad. Sci. URSS (N.S.) **53**, 3–6 (1946).

It is stated without proof that the following result can be proved using methods previously established by the author [same C. R. (N.S.) **44**, 219–222 (1944); these Rev. **6**, 270]. If there exists in Hilbert space $\mathfrak{H}$ (not necessarily complete) a Hermitian operator A and linear, not necessarily continuous, functionals $\Phi_j(f; \lambda)$, $j=1, \cdots, p$; $\lambda \epsilon (-\infty, \infty)$, which depend analytically on λ for every $f \epsilon \mathfrak{H}$, and if $Ag - \lambda_0 g = f_0$ has a solution g_0 in the domain $\mathfrak{D}(A)$ of A if and only if $\Phi_j(f_0, \lambda_0)=0$, $j=1, \cdots, p$, $\lambda_0 \epsilon (-\infty, \infty)$, then there exists a matrix valued function $\mathfrak{T}(\lambda) = \|\tau_{jk}(\lambda)\|$, $j, k=1, \cdots, p$, such that

$$(1) \qquad (g, f) = \sum_{j,k=1}^{p} \int_{-\infty}^{\infty} \Phi_j(g; \lambda)\overline{\Phi_k(f; \lambda)}\tau_{jk}(\lambda)d\lambda$$

for every pair $g, f \epsilon \mathfrak{H}$, and (2) $\sum_{j,k=1}^{p}\tau_{jk}(\lambda)\xi_j\bar{\xi}_k$ is a nondecreasing function of λ for arbitrary complex $\xi_1, \cdots, \xi_p$. If $(Af, f) \geqq 0$ for $f \epsilon \mathfrak{D}(A)$, there exists $\mathfrak{T}(\lambda)$ satisfying (1) and (2) such that $\tau_{jk}(\lambda)=0$, $j, k=1, \cdots, p$, $\lambda \epsilon (-\infty, 0)$. As an application it is indicated how results of the following type can be obtained. A necessary and sufficient condition that $F(x)$, continuous in $(0, a)$, have the form $\int_{-\infty}^{\infty} \cos \lambda^{\frac{1}{2}} x d\tau(\lambda)$, where $\tau(\lambda)$ is a certain nondecreasing function of λ, is that the nucleus (kernel) $F(x+y)+F(|x-y|)$ be positive definite. *R. A. Leibler* (Princeton, N. J.).

Julia, Gaston. Die Funktionentheorie und die Theorie der Operatoren im Hilbertschen Raum. Abh. Preuss. Akad. Wiss. Math.-Nat. Kl. **1942**, no. 11, 15 pp. (1943).

The author outlines a discussion whose purpose is to exploit an analogy between linear transformations in Hilbert space and analytic functions. In this analogy the bounded transformations correspond to integral functions and the closed transformations to those functions with a finite radius of convergence. The comparison is natural enough in convergence questions, and there is an equivalent of the Weierstrass factorization theorem; however, the analogy seems strained in the search for analogues of the notion of analytic extension, where it seems to the reviewer that the theories have essential dissimilarities. It seems to the reviewer, also, that a reader might be led into believing that results obtained by other mathematicians are due to the author. *F. J. Murray* (New York, N. Y.).

Lewitan, B. Integral equations and operations of generalized translation. C. R. (Doklady) Acad. Sci. URSS (N.S.) **51**, 659–661 (1946).

The author uses methods of T. Carleman [Sur les Équations Intégrales Singulières à Noyau Réel et Symétrique, Uppsala, 1923] to study kernels of the form $\bar{T}_t{}^s f(t)$, where T^s is a (noncommutative) generalized translation [Lewitan, same C. R. (N.S.) **47**, 3–6 (1945); these Rev. **7**, 126]. *C. E. Rickart* (New Haven, Conn.).

Lewitan, B. Rings of operators and operations of generalized translation. C. R. (Doklady) Acad. Sci. URSS (N.S.) **52**, 99–101 (1946).

Let B denote the ring of all Hermitian operators A of the form $A\varphi = \int \bar{T}_t{}^s f(t)\varphi(t)dm(s)$, $f, \varphi \epsilon L_2$, where T^s is a generalized translation [Lewitan, same C. R. (N.S.) **47**, 3–6 (1945); these Rev. **7**, 126] and let M be a commutative subring of B. By a result of von Neumann [Math. Ann. **102**, 370–427 (1929)] there is a Hermitian operator R such that

every element of M is a function of R. The author shows that the spectral function of R may be represented in the form $\hat{T}_t{}^s\varphi(t, \lambda)$, where $\varphi(t, \lambda)$ is a continuous function of t ($\lambda \neq 1$) and is of bounded variation with respect to λ in every interval (a, b), $b \neq 1$. *C. E. Rickart.*

Lewitan, B., and Powsner, A. Differential equations of the Sturm-Liouville type on the semi-axis and Plancherel's theorem. C. R. (Doklady) Acad. Sci. URSS (N.S.) **52**, 479–482 (1946).

Consider the equation $u_{xx} - u_{yy} - \{\rho(x) - \rho(y)\}u = 0$, $u(x, 0) = f(x)$, $u_y'(x, 0) = 0$, where $\rho(x)$, $f(x)$ are even. If the solution is written in the form $u(x, y) = T_x{}^y f(x)$, then T^y is a family of linear operators on the functions f. The authors study properties of the operators T^y extended to the space L of integrable functions on $(0, \infty)$. The operator T^y is a generalized translation [Lewitan, same C. R. (N.S.) **47**, 3–6 (1945); these Rev. **7**, 126] and, as a function of y, is strongly continuous in both L and the Hilbert space H of square integrable functions on $(0, \infty)$. It is observed that the general spectral theorem for the ring of operators of the form $A\varphi = \int_0^\infty T_x{}^y f(x)\varphi(y)dy$ ($\varphi \epsilon H$) becomes a theorem of Plancherel type when the spectral function is differentiable.
C. E. Rickart (New Haven, Conn.).

Fichera, Gaetano. Sull'ubicazione e l'unicità delle estremanti del polinomiale quadratico nella sfera di Hilbert. Ist. Naz. Appl. Calcolo (2) no. **160**, 18 pp. (1944).

Fichera, Gaetano. Sull'ubicazione e l'unicità delle estremanti del polinomiale quadratico nella sfera di Hilbert. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **4**, 213–216 (1943).

The author is concerned with the location and uniqueness of extrema associated with the quadratic functional

$$P(\varphi) = \int\int K(x, y)\varphi(x)\varphi(y)dxdy - 2\int f(x)\varphi(x)dx,$$

where K and f are Lebesgue square integrable and K is symmetric. He uses previous existence theorems of Picone and Tonelli [Ann. Mat. Pura Appl. (4) **18**, 1–21 (1939); these Rev. **1**, 77] relating to the extrema of P on the unit sphere in L^2. The author states without proof that, if K is definite, then P has a maximum and minimum in the unit sphere. T~~his result is not true without certain additional hypotheses which the author does not explicitly give.~~ The first of the papers is a note announcing his results, some of which are proved in the second paper. *H. H. Goldstine.*

* For Errata, please see p. 709.

Lévy, Paul. Problème de Dirichlet et surfaces minima dans l'espace de Hilbert. C. R. Acad. Sci. Paris **223**, 772–773 (1946).

Indications regarding improvements on the theory of minimal surfaces in Hilbert space as developed in the author's Leçons d'Analyse Fonctionelle [Paris, 1922], in particular, pages 375–420. The details are to appear in the second edition of this book. *H. Busemann.*

Sebastião e Silva, José. Sull'analisi funzionale lineare nel campo delle funzioni analitiche. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **1**, 709–715 (1946).

The paper summarizes results the author has obtained in modifying the work of Fantappiè [Atti Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. **12**, 617–706 (1942); these Rev. **8**, 158] on the theory of analytic functionals of analytic functions. The space (C) of all functions $f(z)$ locally analytic (for some region R containing the closed set C, f is analytic on R) is replaced by the space $\mathfrak{F}[C]$, in which any function $f(z)$ of (C) together with all its analytic extensions which belong to (C) are considered to be the same function. The vicinity topology is replaced by a sequential convergence topology, i.e., φ_n of $\mathfrak{F}[C]$ converges to φ of $\mathfrak{F}[C]$ if and only if the φ_n and φ have a common region of regularity and $\varphi_n \to \varphi$ uniformly on this region. An analytic functional F on $\mathfrak{F}[C]$ transforms a function $\varphi(\alpha, z)$ into an analytic function of α on some region Ω of the complex α-sphere, $\varphi(\alpha, z)$ being analytic in α in the sense that, for a vicinity of each α_0 of Ω, $\varphi(\alpha, z) = \lim_n \sum_{j=1}^n (\alpha - \alpha_0)^i \varphi_i(z)$, where $\varphi_i(z)$ belong to $\mathfrak{F}[C]$. Then an analytic distributive functional and a linear continuous functional on $\mathfrak{F}[C]$ are identical and take the Fantappiè form $F(\varphi(z)) = (2\pi i)^{-1}\int_\gamma \varphi(\alpha)u(\alpha)d\alpha$, where $u(\alpha) = F_z(1/(\alpha - z))$ is regular in the complement of C and γ is a convenient boundary of a closed region containing C on which $\varphi(\alpha)$ is regular. The expression

$$\varphi(F) = (2\pi i)^{-1}\int_\gamma \frac{\varphi(\alpha)d\alpha}{\alpha I - F}$$

is given, relating functions $\varphi(z)$ of $\mathfrak{F}[C]$ with functions $\varphi(F)$ of the distributive transformation F on a vector space S into itself, the set C being the closed bounded set for which $(\alpha I - F)^{-1}$ does not exist [see Dunford, Trans. Amer. Math. Soc. **54**, 185–217 (1943); these Rev. **5**, 39].
T. H. Hildebrandt (Ann Arbor, Mich.).

Amerio, Luigi. Una metrica per lo spazio delle funzioni misurabili. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **3**, 343–349 (1942).

The author considers the space Q of measurable functions defined over a measurable set K (finite or infinite) of n-dimensional Euclidean space. He shows that a metric can be introduced into this space so that it is complete and separable relative to this metric. To define the metric he introduces the notion of the weight of a subset J of K as $\int_J F(Z)dx_1 \cdots dx_n$, where F is a positive function defined on K and is such that the weight of K itself is finite. (In the integral above, $Z = (x_1, \cdots, x_n)$ is an arbitrary point of K.) If $f_1(Z)$, $f_2(Z)$ correspond to two points P_1 and P_2 of Q and p is nonnegative, let μ_p be the weight of the set for which $|f_1(Z) - f_2(Z)| \geqq p$. Then it is shown that μ_p is nonincreasing. The lower bound of all p for which $p - \mu_p > 0$ is taken to be the distance from P_1 to P_2. The author closes by indicating how the usual axioms are satisfied by this metric. *H. H. Goldstine* (Princeton, N. J.).

Scorza Dragoni, Giuseppe. Un teorema d'esistenza per gli elementi uniti di una trasformazione funzionale. Rend. Sem. Mat. Univ. Padova **15**, 25–32 (1946).

Let Σ be the space of functions $\varphi(x)$ of class $C^{(n)}$ with distance given by $\max \sum_{i=0}^n |\varphi^{(i)}(x) - \psi^{(i)}(x)|$ on $-1 \leqq x \leqq 1$. The author combines a theorem of the Birkhoff-Kellogg-Schauder type with one due to Cacciopoli [same Rend. **3**, 1–15 (1932)] to obtain the following theorem. If F is on Σ to Σ and of the form $F(\varphi) = S(\varphi) + T(\varphi)$, where S is continuous on Σ and transforms Σ into a compact subset of Σ, while the transformation $\varphi - T\varphi$ is continuous, locally invertible and transforms a compact sequence into a convergent sequence, then F has an invariant element. The special case when Σ is n-dimensional Euclidean space is proved directly by using the theorem of Rouché [Alexandroff and Hopf, Topologie I, Springer, Berlin, 1935, p. 459].
T. H. Hildebrandt (Ann Arbor, Mich.).

Delange, Hubert, et Pauc, Christian. L'extensibilité des espaces vectoriels normés. C. R. Acad. Sci. Paris **223**, 606–608 (1946).

(I) If V is a normed vector space and S is any finite subset of V, write $\delta^*(S)$ for the maximum, for $T\subset S$, of $\|\sum v\|$, $v\varepsilon T$; $\lambda(S)=\sum\|v\|$ for $v\varepsilon S$, and $\rho(v)=\inf_S\{\delta^*(S)/\lambda(S)\}$. A necessary and sufficient condition that every unconditionally convergent series in V is absolutely convergent is that $\rho(V)$ is positive. (II) Similarly, if S is a rectifiable curve (of length $\lambda(S)<\infty$) in V, consider the mass distribution on subsets B of the unit sphere in V defined as the linear measure on S of the set of points at which the oriented unit tangent vector belongs to B. Two curves are called equivalent if they define the same mass distribution; $\delta^*(S)$ is defined as the supremum of the diameters of curves equivalent to S. If V is uniformly convex, $\delta^*(S)\geqq\rho(V)\lambda(S)$. (III) Finally, if m is a measure on a σ-field $\mathfrak{K}$ with values in V and $K\varepsilon\mathfrak{K}$, write $\delta^*(K)=\sup\{\|m(K')\|:K'\subset K,\ K'\varepsilon\mathfrak{K}\}$, and $\lambda(K)=\sup\{\sum\|m(K_i)\|\}$ for all countable partitions $\{K_i\}$ of K ($K_i\varepsilon\mathfrak{K}$, $i=1,2,\cdots$). Then $\delta^*(K)\geqq\rho(V)\lambda(K)$. The authors remark that they know of no nontrivial (infinite dimensional) example of a space V for which $\rho(V)$ is positive. No proofs are given. *P. R. Halmos* (Chicago, Ill.).

Eberlein, William F. Closure, convexity, and linearity in Banach spaces. Ann. of Math. (2) **47**, 688–703 (1946).

A positive homogeneous functional $K(x)$ is called convex if its domain D_K is a linear subspace of a Banach space B and it satisfies the triangle inequality. Let $D^0=\{x\,|\,K(x)=\theta\}$; then the natural homomorphism of D_K to D_K/D^0 defines a distributive transformation T from B_1 to B_2, the complete extension of D_K/D^0, and $K(x)=\|T(x)\|$. (Obviously for any distributive T from a B_1 to a B_2, $\|T(x)\|$ is convex.) This is a fundamental representation, (A). The author introduces a "partially convex" set, i.e. x, $-x$, $\frac{1}{2}(x+y)$ are members if x and y are. Using this concept, a slight modification of a proof in Banach's book establishes that, if $K(x)$ is convex and D_K is of the second category and has the Baire property, then $K(x)$ is continuous. The advance lies in the elimination of a lower semi-continuity condition on $K(x)$. It is shown that a distributive transformation is closed (lower semi-continuous) if the set of x for which $\|T(x)\|\leqq1$ is closed in B_1 (in D_T). A lower semi-continuous transformation admits a closed extension. As expected, the converse is true if the unit sphere of B_2 is $w(B_2)$ sequentially compact. Moreover, with this last condition, the power (assumed nonfinite) of a dense set in R_T, the range of a distributive closed T, does not exceed that of a dense set in D_T.

In a (not necessarily separable) Hilbert space a convex $K(x)$ is required to satisfy (P) the well-known Jordan-von Neumann condition on the norms. The results here depend on the possibility of other representations than (A). Thus if T is closed, with domain dense in H, then (B) $K(x)=\|T(x)\|=\|S(x)\|$, $x\varepsilon D_T=D_S$, where $S=(T^*T)^{\frac{1}{2}}$ is self-adjoint. If T is positive definite, $K(x)=(Tx,x)^{\frac{1}{2}}$ is lower semi-continuous. This observation gives a simple proof of the essentially known theorem that a distributive, symmetric T from H to H with an upper (lower) bound admits a self-adjoint extension S of the type in (B) with the same bound. The last part of the paper is concerned with the question of when D is a possible domain of a self-adjoint transformation. Necessary and sufficient conditions are that D is dense in H and is the extension of a closed, convex, bounded, symmetric set whose Minkowski functional satisfies (P). All proofs in this paper are quite direct. *D. G. Bourgin* (Urbana, Ill.).

Kondô, Motokiti. La produit kroneckerienne infinie des espaces linéaires. Proc. Imp. Acad. Tokyo **20**, 569–579 (1944). [MF 14927]

Guided by von Neumann's infinite direct product for unitary spaces [Compositio Math. **6**, 1–77 (1938)] the author constructs a direct product for an infinite class C of Banach spaces. He assumes that for every finite subset of C a suitable direct product is given, having the cross-norm property $\|f_1\otimes\cdots\otimes f_n\|=\|f_1\|\cdots\|f_n\|$; von Neumann's equivalence of infinite vector products is then expressed in terms of the metric in these finite direct products. This determines uniquely a generalization of the incomplete direct product. A direct sum of these incomplete products is defined to be the complete direct product. Brief comments are made on the direct products of conjugate spaces and operator rings. The direct product of the conjugate spaces, assuming that the associated norm also has the cross-norm property, is stated to be identical with the conjugate of the direct product. This is not true in general, as shown by Dunford and Schatten [Trans. Amer. Math. Soc. **59**, 430–436 (1946); these Rev. **7**, 455]. The author's construction of an L_p product of Banach spaces is not valid, even for finite products, except when $p=1$. *I. Halperin.*

Izumi, Shin-ichi. Notes on Banach space. I. Differentiation of abstract functions. Proc. Imp. Acad. Tokyo **18**, 127–130 (1942). [MF 14744]

Let R_0 be a Euclidean figure and x_s a function on points $s\varepsilon R_0$ to a Banach space X. In X let Y be the linear span of the functional values of x_s, and say for a sequence $\{\gamma_n\}$ in X^* that $\{\gamma_n\}\varepsilon N(Y)$ if $\|\gamma_n\|=1$ and $y\varepsilon Y$ imply $\|y\|=\lim\sup\|\gamma_n(y)\|$. The author generalizes theorems of Gelfand and the reviewer by proving that: (1) if $\{\gamma_n\}\varepsilon N(Y)$ and $\gamma_n(x_s)$ is measurable then x_s is measurable in the sense of Jeffery [Duke Math. J. **6**, 706–718 (1940); these Rev. **2**, 103]; (2) if $X(R)$ is defined to X from figures in R_0, and $\gamma_n(X(R))$ is differentiable a.e. to $\gamma_n(x_s)$, where $\{\gamma_n\}\varepsilon N(Y)$, then x_s is Jeffery-measurable, and if $X(R)$ is of bounded variation (BV) then $\|x_s\|$ is summable and hence x_s is Birkhoff-integrable. The author is then able to remove from some theorems of the reviewer [Duke Math. J. **5**, 254–269 (1939)] the assumption that Y is separable; e.g., if $X(R)$ is additive and BV and there exists $\{\gamma_n\}\varepsilon N(X)$ such that $\gamma_n(X(R))$ is differentiable a.e. to $\gamma_n(x_s)$ then x_s is Birkhoff-integrable, and if either $X(R)$ or $\int_R x_s ds$ is differentiable a.e. then both must be and both derivatives must equal x_s a.e. Included are brief remarks concerning applications to abstract fractional differentiation and convolution kernels. In the proofs of theorems 2.1 and 2.2 [(1) and (2) above] references to pages 714 and 713 of Jeffery's paper seem required. *B. J. Pettis* (New Haven, Conn.).

Arens, Richard. On a theorem of Gelfand and Neumark. Proc. Nat. Acad. Sci. U. S. A. **32**, 237–239 (1946).

Gelfand and Neumark proved [Rec. Math. [Mat. Sbornik] N.S. **12**(54), 197–213 (1943); these Rev. **5**, 147] that an Abelian normed $*$-ring is the continuous functions on a compact space. This paper simplifies the proof, eliminating the use of an inaccessible paper of Gelfand, Raikov and Šilov. *W. Ambrose* (Princeton, N. J.).

Riesz, Frédéric. Sur quelques problèmes de la théorie ergodique. Mat. Fiz. Lapok **49**, 34–62 (1942). (Hungarian. French summary)

Except for slight changes in a few proofs this paper is identical with the author's subsequent Geneva address [Comment. Math. Helv. 17, 221–239 (1945); these Rev. **7**, 255]. *P. R. Halmos* (Chicago, Ill.).

Dunford, Nelson, and Miller, D. S. On the ergodic theorem. Trans. Amer. Math. Soc. **60**, 538–549 (1946).

Let S be a measure space of finite measure, t a variable point in S and $L(S)$ the Lebesgue space of real summable functions on S. The authors consider transformations of the type $Tf=f(\varphi t)$ arising from transformations φ, of S into itself, which are not necessarily measure preserving. Instead the following weaker conditions are assumed: (a) φ does not transform nonmeasurable sets into measurable sets, (b) φ does not transform sets of positive measure into zero sets. Under these hypotheses it is proved that, for $f\epsilon L(S)$, convergence in $L(S)$ of $n^{-1}\sum_{\nu=0}^{n-1}T^\nu f$ implies pointwise convergence of $n^{-1}\sum_{\nu=0}^{n-1}f(\varphi^\nu t)$ almost everywhere, and is equivalent to the existence of a constant M such that $n^{-1}\sum_{\nu=0}^{n-1}|\varphi^{-\nu}e|\leqq M|e|$ for every measurable set e in S and $n=1, 2, \cdots$. Similar results are obtained for measurable semi-groups of transformations in S. *R. A. Leibler*.

Tsuji, Masatsugu. On Hopf's ergodic theorem. Proc. Imp. Acad. Tokyo **20**, 640–647 (1944). [MF 14936]

Tsuji, Masatsugu. Some metrical theorems on Fuchsian groups. Proc. Imp. Acad. Tokyo **21**, 104–109 (1945). [MF 14955]

Tsuji, Masatsugu. On Hopf's ergodic theorem. Jap. J. Math. **19**, 259–284 (1945). [MF 15005]

The third paper contains the results of the first two, with proofs given in expanded form. Let G be a Fuchsian group with principal circle U, $|z|=1$, and let R_0 be the usual fundamental region of G. If $u=e^{i\theta}$, $v=e^{i\phi}$ are arbitrary points on U, the pair (u, v) can be considered as a point on a torus T and the group G defines a group G_T on T. Measure is defined on T by product measure and consequently $mT=4\pi^2$. Let $z_1, z_2, \cdots$ be the points congruent to $z_0=0$ under G and let $n(r)$ denote the number (necessarily finite) of these points in the circle $|z|=r<1$. The author proves as his main theorem the following result: if

$$\limsup_{r\to 1} n(r)(1-r)>0$$

then there does not exist a measurable set E on T such that $0<mE<4\pi^2$. The latter condition is equivalent to metric transitivity of the flow defined by the hyperbolic lines on the manifold obtained by identifying points which are congruent under G. The author states that his theorem is an extension of a theorem of E. Hopf [Trans. Amer. Math. Soc. **39**, 299–314 (1936)] and, as Hopf did, the author makes elaborate use of harmonic functions.

It may be remarked that, by use of a more recent paper of Hopf [Ber. Verh. Sächs. Akad. Wiss. Leipzig **91**, 261–304 (1939); these Rev. 1, 243] and one of the author's previous theorems which he cites, the main theorem can be proved briefly. Let $F(\theta)$ be the points of R_0 which are congruent to the points on the radius $z=re^{i\theta}$ $(0\leqq r<1)$. The author quotes as a previous result: if $\sum_{n=0}^{\infty}(1-|z_n|)=\infty$ then $F(\theta)$ is everywhere dense in R_0 for almost all $e^{i\theta}$ on $z=1$. Now $\limsup_{r\to 1} n(r)(1-r)>0$ implies $\sum_{n=0}^{\infty}(1-|z_n|)=\infty$ and, according to Hopf, if $F(\theta)$ is everywhere dense in R_0 for almost all $e^{i\theta}$ on $z=1$, we have metric transitivity, which implies that there cannot exist a measurable set E on T such that $0<mE<4\pi^2$. Thus the main theorem is proved.

There are some additional results of an elementary nature concerning the flow defined by the hyperbolic lines.

G. A. Hedlund (Charlottesville, Va.).

Niemytzki, V. Les systèmes dynamiques généraux. C. R. (Doklady) Acad. Sci. URSS (N.S.) **53**, 491–494 (1946).

Let X be a complete metric space and let G be a locally compact topological group, satisfying the second countability axiom, of homeomorphisms of X onto X such that (1) $g_1(g_2(x))=(g_1g_2)(x)$, (2) $g(x)$ defines a continuous transformation of the product space $X\times G$ onto X, (3) if e is the unit element in G, $e(x)=x$ for all x in X and e is the only element in G with this property. Like a recent note by Barbachine [same C. R. **51**, 3–5 (1946); these Rev. **8**, 34], the present paper generalizes certain aspects of Birkhoffian dynamics (in which case G is the real axis) to the general case under consideration. If $x\epsilon X$, the set $\Omega(x)$ is the set of all points y of X such that corresponding to y there is a sequence $g_1, g_2, \cdots$ of elements of G such that this sequence has no limit points in G and $g_n(x)\to y$. The point x is stable in the sense of Poisson if, $O(x)$ denoting the orbit of x, $\Omega(x)\cap O(x)\neq\phi$ and $\overline{O(x)}-O(x)\neq\phi$. It is shown that, if x is stable in the sense of Poisson, then in any neighborhood of any point of $O(x)$ there exist points which are not in $O(x)$. A point x is said to be recurrent if corresponding to $\epsilon>0$ there exists a compact set W in G such that $e\epsilon W$ and, if y is any point of $O(x)$, the set $f(y, W)$ (that is, the set of all $g(y)$ for which $g\epsilon W$) is ϵ-dense in $O(x)$. It is proved that, if M is a compact invariant subset of X such that the orbit of every point of M is everywhere dense in M, then each point of M is recurrent. As remarked by the author, this result can be obtained from those of Barbachine. It is also implied by a theorem due to Gottschalk [Ann. of Math. (2) **47**, 762–766 (1946), theorem 5; these Rev. **8**, 159].

G. A. Hedlund (Charlottesville, Va.).

Theory of Probability

Nilssen, Bailli. Some remarks on counting. Norsk Mat. Tidsskr. **27**, 106–111 (1945). (Norwegian)

By means of recursion formulas the author computes certain probabilities connected with placing balls in compartments. He also computes the number of ways in which a number can be written as a sum of n numbers with the order of the terms taken into account. These problems are said to be connected with corrections for automatic counters.

W. Feller (Ithaca, N. Y.).

Ostrowski, Alexandre. Sur la formule de Moivre-Laplace. C. R. Acad. Sci. Paris **223**, 1090–1092 (1946).

The author gives an estimate of the error term in the normal approximation to the binomial distribution. It appears that his result is weaker than that of Uspensky [Introduction to Mathematical Probability, McGraw-Hill, New York, 1937, pp. 129 ff.] of which he seems unaware. The proof is sketched but the details are to appear elsewhere.

W. Feller (Ithaca, N. Y.).

Borel, Émile. Sur les probabilités dénombrables et le pari de Pascal. C. R. Acad. Sci. Paris **224**, 77–78 (1947).

In the author's opinion the following computation of an expected value leads to an expression of the form $0\cdot\infty$ with

true value 0. Let p_k $(k=0, 1, \cdots, \infty)$ be the probability that k among the events $\{A_k\}$ are realized. The expected number of events realized is said to be $p_1+2p_2+\cdots+\infty p_\infty$; however, if it is known that with probability one only a finite number of events materialize, the same expectation is said to be $p_1+2p_2+\cdots$, whence $\infty \cdot p_\infty = 0$.

W. Feller (Ithaca, N. Y.).

Gotusso, Guido. Probabilità di rottura di un filo. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) **9(78)**, 182–190 (1945).

Let $f(p, l)$ be the probability that a thread of length l withstands a force p. Assuming statistical independence between sections of the threads one gets $f(p, l_1+l_2)=f(p, l_1)f(p, l_2)$, whence $f(p, l)=(\psi(p))^l$. For $\psi(p)$ the author suggests a normal distribution.

W. Feller (Ithaca, N. Y.).

Eyraud, H. Sur l'addition des aléatoires imaginaires. Ann. Univ. Lyon. Sect. A. (3) **2**, 7–17 (1939).

The author considers chance variables which take on complex values and indicates simple methods for finding those which (a) take on a finite number of values and have given moments, (b) are linear combinations of independent chance variables with Poisson distributions and have given moments.

J. L. Doob (Urbana, Ill.).

Stein, Charles. A note on cumulative sums. Ann. Math. Statistics **17**, 498–499 (1946).

Let $\{Z_i\}$ be a sequence of mutually independent random variables with the same distribution. For any two given constants a, b with $b<0<a$, define the random variable n as the smallest integer for which one of the inequalities $\sum_1^n Z_i \geqq a$ or $\sum_1^n Z_i \leqq b$ holds. It is shown that there exists a t_0 such that the moment generating function $E(e^{nt})$ exists for all t with $\Re t \leqq t_0$. It follows, in particular, that n has finite moments of all orders. For the proof the author shows by elementary considerations that there exists a q with $0<q<1$ such that $\Pr\{n \geqq m\} < q^m$ for $m=1, 2, \cdots$.

W. Feller (Ithaca, N. Y.).

Robbins, Herbert. On the $(C, 1)$ summability of certain random sequences. Bull. Amer. Math. Soc. **52**, 699–703 (1946).

Simple corollaries of the strong law of large numbers.

M. Kac (Ithaca, N. Y.).

Votaw, David F., Jr. The probability distribution of the measure of a random linear set. Ann. Math. Statistics **17**, 240–244 (1946).

Let $X_1, \cdots, X_n$ be independent random variables having the same distribution function $F(x)$. Consider the set consisting of the intervals $(X_\nu-\frac{1}{2}D, X_\nu+\frac{1}{2}D)$ and denote its measure by S. The author calculates the distribution function of S in the case the X's are uniformly distributed and the characteristic function of the distribution function of S in the case

$$F(x)=\int_0^x He^{-Ht}dt,\ x\geqq 0;\ F(x)=0,\ x<0.$$

M. Kac (Ithaca, N. Y.).

Offord, A. C. An inequality for sums of independent random variables. Proc. London Math. Soc. (2) **48**, 467–477 (1945).

Let $X_1, \cdots, X_n$ be independent random variables and let

$$\alpha_\nu = E\{X_\nu\},\quad A_\nu^2 = E\{(X_\nu-\alpha_\nu)^2\},\quad B_\nu^3 = E\{|X_\nu-\alpha_\nu|^3\}.$$

Let furthermore $F_n(u)$ be the distribution function of $X_1+\cdots+X_n$ and

$$\psi_n(x) = \max_{-\infty<t<\infty} \int_{t-x}^{t+x} dF_n(u).$$

The main result of the paper is the following useful inequality:

$$\psi_n(x) \leqq C\kappa^{-2}n^{-\frac{1}{2}} \log n \{\log n + \kappa x/\min_\nu A_\nu\};$$

C is an absolute constant and κ is defined by the formula $2\kappa^{\frac{1}{2}} = \min_\nu A_\nu/B_\nu$.

M. Kac (Ithaca, N. Y.).

Domb, C. The resultant of a large number of events of random phase. Proc. Cambridge Philos. Soc. **42**, 245–249 (1946).

Limiting distributions of sums of independent random variables of special types are found by solving appropriate parabolic differential equations (this method, as the author points out, goes back to Rayleigh). The derivation of the differential equations is purely heuristic. [The author's results can be obtained with equal simplicity and with complete rigor using the technique of characteristic functions.]

M. Kac (Ithaca, N. Y.).

Pitt, H. R. A theorem on random functions with applications to a theory of provisioning. J. London Math. Soc. **21**, 16–22 (1946).

The author considers a stock which is being consumed at a rate c (Poisson distribution). It is supposed that there is an initial stock N and two methods of replacement are compared. (A) Replacements are ordered at irregular times, in amounts of a fixed size P, whenever P units have been consumed; (B) replacements are ordered at regular intervals, of length P/c, the amount ordered being the consumption during the previous interval. In both cases the time between orders has mean value P/c and the order has mean value P. It is shown that the replacement system (A) is more advantageous than (B) in the sense that (roughly speaking) for a given mean positive stock the mean negative stock is numerically larger for (B) than for (A).

J. L. Doob (Urbana, Ill.).

Loève, Michel. Fonctions aléatoires de second ordre. Revue Sci. **84**, 195–206 (1946).

Expository article.

Natta, G. Leggi di ripartizione delle singole specie molecolari nei prodotti di una catena di reazioni successive. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) **9(78)**, 307–320 (1945).

The author solves the familiar system of differential equations $y_0' = -a_0y_0$, $y_n' = -a_ny_n + a_{n-1}y_{n-1}$, with a_n = constant. He then considers the case $a_n = c_n \cdot f(t)$ with c_n = constant, which reduces to the former if a new time parameter $\tau = \int f(t)dt$ is introduced. The author's formulas contain $tf(t)$ instead of this τ.

W. Feller (Ithaca, N. Y.).

Popoff, Cirillo. Osservazioni sulla teoria delle probabilità concatenate di Markoff. Caso di una successione continua di prove. Univ. Roma e Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **3**, 282–292 (1942).

It is shown that for Markov chains with a finite number of states and depending on a continuous time parameter the ergodic principle holds.

W. Feller (Ithaca, N. Y.).

Cunningham, L. B. C., and Hynd, W. R. B. Random processes in problems of air warfare. Suppl. J. Roy. Statist. Soc. **8**, 62–85 (4 plates) (1946).

Several applications of the theory of stochastic processes are described of which the following is typical. In anti-aircraft gunnery an automatic predictor predicts the position of the target at any future date. The prediction is based on data, provided by a radar set, where random errors are superimposed upon the true values. The various harmonic components of the error are magnified by factors which are functions of their frequencies. To select an optimum predictor the error spectrum of the particular radar set must be investigated. In normal practice the errors are smoothed, but an analysis shows that smoothing may introduce large new errors and that only high frequency components of the tracking errors can be practically eliminated. In the analysis the predictor is considered as a linear device and is characterized by its response to the so-called Heaviside impulse function. The spectrum of a random process is obtained in the now usual way from the Fourier transform of the correlogram.

The last section [pp. 81–85] describes a relay-machine constructed for the Ministry of Aircraft Production (designed and constructed by Weir and Barnes) which evaluates scalar products $\sum x_k y_k$. The data have to be grouped together to integer values in the range -63 to 63 (-24 to 24 for an older machine by Shire and Runcorn). The machine comprises six main parts: a keyboard perforator, two transmitters, a lamp display, a control box, a computer rack. The two sequences of integers x_k and y_k are recorded in code as perforations on each of two tapes, which are then read automatically by two transmitters. The machine then performs the computations, the result being shown on the lamp display. The machine is also useful in evaluating approximately integrals of the form $\int_a^b f(x)g(x)dx$, in particular, Fourier coefficients. [The description refers to the plates, which appear opposite p. 48.] Pages 85–97 contain a discussion of this paper and two others. *W. Feller.*

Hole, Njål. On the statistical treatment of counting experiments in nuclear physics. Ark. Mat. Astr. Fys. **33A**, no. 11, 11 pp. (1946).

A new derivation of the time distribution of counts with Geiger-Müller and similar counters with finite resolving time. The last section treats the case of two counters in series. *W. Feller* (Ithaca, N. Y.).

Pollaczek, Félix. Sur quelques lois asymptotiques de la théorie de l'encombrement des réseaux téléphoniques. Ann. Univ. Lyon. Sect. A. (3) **5**, 21–35 (1942).

Simple asymptotic expressions are given for the probability of a delay when there are ρ lines with unlimited access and the holding times are exponentially distributed. The results are compared with the case of constant holding times and with the probability of lost calls in arrangements without waiting time. *W. Feller* (Ithaca, N. Y.).

Mathematical Statistics

Tweedie, M. C. K. Functions of a statistical variate with given means, with special reference to Laplacian distributions. Proc. Cambridge Philos. Soc. **43**, 41–49 (1947).

Let X be a random variable whose distribution depends on a parameter α. It is assumed that either X has a probability density or else X is discrete; in the former case $p(x, \alpha)$ denotes the density, in the latter it denotes the probability that $X=x$. The main problem considered is the existence, uniqueness and determination of a function $u(x)$ such that the expected value of $u(X)$ equals a specified function $U(\alpha)$. A formal solution for $u(x)$ is obtained under heavy restrictions. Most of the paper is devoted to an important class of distributions to which the formal theory is applicable and which the author proposes to call "Laplacian," namely, those for which $p(x, \alpha)$ may be brought into the form $f(x) \exp\{-\alpha x - F(\alpha)\}$ by transformation of the parameter. *H. Scheffé* (Berkeley, Calif.).

Weichelt, John A. A first-order method for estimating correlation coefficients. Psychometrika **11**, 215–221 (1946).

Waugh, Frederick V. The computation of partial correlation coefficients. J. Amer. Statist. Assoc. **41**, 543–546 (1946).

Marcantoni, Alessandro. Il principio dei minimi quadrati. Univ. Roma e Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **3**, 192–202 (1942).

Elementary remarks on the estimation of the mean of a normal distribution by the method of least squares from n observations and the determination, by Bayes's formula, of the distribution a posteriori of the error. *W. Feller.*

Eyraud, H. Les lois d'erreurs dans deux dimensions. Ann. Univ. Lyon. Sect. A. (3) **2**, 19–23 (1939).

If in estimating the center (m_1, m_2) of a two-dimensional distribution, with density $\varphi(x-m_1, y-m_2)$, from a sample of n independent observations, the maximum likelihood estimate coincides with the estimate obtained by minimizing $\sum_j\{(x_j-m_1)^2+(y_j-m_2)^2\}^{p/2}$, φ must be of the following type: if $p=2$, $\varphi(x, y)$ is normal; if $p\neq 2$, $\log \varphi = c_1+c_2(x^2+y^2)^{p/2}$. *J. L. Doob* (Urbana, Ill.).

Feraud, L. Problème d'analyse statistique à plusieurs variables. Ann. Univ. Lyon. Sect. A. (3) **5**, 42–53 (1942).

An expository lecture.

Bhattacharyya, A. On a measure of divergence between two multinomial populations. Sankhyā **7**, 401–406 (1946).

This is a preliminary article to one previously reviewed [Bull. Calcutta Math. Soc. **35**, 99–109 (1943); these Rev. **6**, 7]. The divergence between two multinomial populations with probabilities $\{\pi_i\}$ and $\{\pi_i'\}$ ($\sum \pi_i = \sum \pi_i' = 1, i=1, \cdots, k$) is defined by $\cos \Delta = \sum(\pi_i \pi_i')^{\frac{1}{2}}$. Let D be the estimate of Δ for samples of n and n'. If $n'=n$ and $\Delta=0$, $D=\chi/(2n)^{\frac{1}{2}}$ for large n. For $\Delta\neq 0$, the density function of D is proportional to $D^{(k-1)/2}e^{-c(D^2+\Delta^2)}I_{(k-3)/2}(2cD\Delta)$, where $c=2nn'/(n+n')$. Here n' need not equal n. A first approximation to D^2 for small Δ is $\sum(p_i-p_i')^2/(p_i+p_i')$. To study the divergence of a sample estimate β from Δ, $\beta=\chi/2n^{\frac{1}{2}}$ for large n. *R. L. Anderson* (Raleigh, N. C.).

Sun, Shu Peng. On the successive approximation to the distribution of the third moment about the mean of independent variates. Acad. Sinica Science Record **1**, 351–354 (1945).

This note contains an asymptotic expansion of the distribution of the third sample moment about the mean, giving also an estimate of the order of magnitude of the remainder term. The author states that the proofs are too lengthy to include in the note. *A. Wald.*

Lehmann, Erich. Une propriété optimale de certains ensembles critiques du type *A*. C. R. Acad. Sci. Paris **223**, 567–569 (1946).

Let the distribution of the sample E be given by an elementary probability law of the form

$$p(E|\theta)=\exp\{P(\theta)+T(E)Q(\theta)+R(E)\}$$

involving an unknown parameter θ. Let, furthermore, W be a critical region of type A_1 and of size ϵ for testing the hypothesis that θ is equal to a specified value θ_0. Finally, let W^* be any other critical region of size ϵ for testing the same hypothesis. It is shown that, if $\theta_1<\theta_0<\theta_2$ and if the power of W^* exceeds that of W with respect to one of the two alternatives θ_1 and θ_2, then the power of W^* falls short of the power of W with respect to the other alternative.

A. Wald (New York, N. Y.).

Chung, Kai-Lai. The approximate distribution of Student's statistic. Ann. Math. Statistics **17**, 447–465 (1946).

The distribution function of the mean of a sample of n independent and equidistributed random variables admits, for large n, an asymptotic expansion in powers of $n^{-\frac{1}{2}}$ first obtained by the reviewer. The method was considerably simplified by P. L. Hsu [same Ann. **16**, 1–29 (1945); these Rev. **6**, 233], who also obtained a similar expansion for the sample variance. In the present paper the method is extended to the Student statistic of the sample. Suppose that the independent variables $x_1, \cdots, x_n$ all have the same distribution with zero mean and unit variance, and write $\bar{x}=\sum_1^n x_i/n$, $s^2=\sum_1^n(x_i-\bar{x})^2/n$, $F(z)=\Pr(n^{\frac{1}{2}}\bar{x}/s\leqq z)$. Then it is shown that, subject to certain general conditions,

$$F(z)-(2\pi)^{-\frac{1}{2}}\int_{-\infty}^{z} e^{-t^2/2}dt$$

admits an asymptotic expansion in powers of $n^{-\frac{1}{2}}$. Explicit expressions are given for the first terms of the expansion.

H. Cramér (New Haven, Conn.).

Pillai, K. C. S. Confidence interval for the correlation coefficient. Sankhyā **7**, 415–422 (1946).

Let r be the correlation coefficient calculated from a sample of n drawn from a bivariate normal population with correlation coefficient ρ. The author considers tests of the hypothesis $\rho=\rho_0$ based on the use of the statistic r. The main interest of the paper lies in a new transformation leading to a simple approximation to the distribution of r. The transformation, suggested by U. S. Nair, is $x=(\rho-r)/(1-\rho r)$. For $-1\leqq x\leqq 1$ the probability density of x is approximated as $c(1-x^2)^{\frac{1}{2}(n-4)}$ plus an odd function of x which may be ignored in calculating the probability that x falls in a symmetrical interval $-a\leqq x\leqq a$. The use of such symmetrical intervals leads to tests whose bias is very small. The calculation of probabilities based on this approximation may be made from incomplete beta or t-tables. For the symmetrical intervals the results appear to be more accurate than those obtained by Fisher's transformation from r to an approximately normal variable [this claim is based on a table containing some obvious misprints].

H. Scheffé (Berkeley, Calif.).

Nandi, H. K. Note on tests applied to samples from normal bivariate population. Science and Culture **12**, 249 (1946).

Bartlett, M. S. On the theoretical specification and sampling properties of autocorrelated time-series. Suppl. J. Roy. Statist. Soc. **8**, 27–41 (1946).

Stationary time series $x(t)$, especially the sampling properties of their autocorrelation coefficients r_u, have been studied independently in many research centers in recent years. The author compares the work of the English school with other contributions and adds new results. The discussion in § 2 sheds light on M. G. Kendall's experimental findings about the r_u [cf. Biometrika **33**, 105–122 (1944); these Rev. **6**, 163]; Bartlett here makes use of a general formula, due to E. Slutsky, for the sampling variance of r_u [although Bartlett regrets that he can give only a second-hand reference to a Russian paper of 1929, Slutsky states the same formula in Giorn. Ist. Ital. Attuari **5**, 435–482 (1934), in particular, p. 471]. In §§ 3–6 Bartlett compares the differential and the finite difference theories of $x(t)$ [cf. also J. L. Doob, Ann. Math. Statistics **15**, 229–282 (1944); these Rev. **6**, 89], studying especially the consequential and largely unexplored fact that, if the analysis is based on non-continuous (say integral) t-values, there may be a loss of accuracy and efficiency. In §§ 3–4 he treats the estimation of the "true" value of r_u, say ρ_u [the statement, p. 33, that an $x(t)$ with integral t cannot be extended to all t if $\rho_u=\rho^u$ $(-1<\rho<0,\ u=0, 1, 2, \cdots)$ is not covered by Bartlett's reference to the reviewer, and is actually untrue]; § 5 deals with series $x(t)$ defined by $x''(t)+\alpha_1x'(t)+\alpha_2x(t)=z(t)$, § 6 with the estimation of the parameters α_i, one of the results being that under special conditions the efficiency of an estimate based on r_1 and r_2 cannot be improved by using instead other r_h and r_k. [Pp. 85–97 contain a discussion of this paper and two others.]

H. Wold (Uppsala).

Bartlett, M. S. A modified probit technique for small probabilities. Suppl. J. Roy. Statist. Soc. **8**, 113–117 (1946).

The probability of an event is assumed to be given by $N(cx+d)$, where $N(t)$ is the cumulative normal distribution (normalized error integral), x is a controllable quantity, c and d are unknown constants. An analysis is given for a testing procedure where tests are made at a given level until the event occurs twice, and this is then repeated at a new level. The quantity estimated is the value of x at which $cx+d$ equals a preassigned value. The analysis is based on large sample weights in the fitting of a straight line. Tables of the appropriate weights are given. The practical advantages of the method are discussed in comparison to the usual probit procedure.

J. W. Tukey (Princeton, N. J.).

Wald, Abraham. Some improvements in setting limits for the expected number of observations required by a sequential probability ratio test. Ann. Math. Statistics **17**, 466–474 (1946).

In the author's paper on sequential analysis [same Ann. **15**, 283–296 (1944); these Rev. **6**, 88], upper and lower limits for the expected number n of observations required by a sequential probability ratio test were derived. Those limits diverge when the expected value of z (a term in the cumulative sum) is near zero. The present paper obtains upper and lower bounds for $E(n)$ which are close together when $E(z)$ is in the neighborhood of zero. The limits are expressed in terms of limits for the expected values of certain functions of Z_n, the cumulative sum. The specific results do not lend themselves to brief recapitulation.

A. M. Mood (Ames, Iowa).

Wald, Abraham. Differentiation under the expectation sign in the fundamental identity of sequential analysis. Ann. Math. Statistics 17, 493–497 (1946).

The author shows that his fundamental identity $E[e^{Z_n t}\varphi(t)^{-n}]=1$ [same Ann. 15, 283–296 (1944); these Rev. 6, 88] may be differentiated any number of times under the expectation sign at any value of t such that $\varphi(t)\geqq 1$ provided $\varphi(t)$ exists for all real values of t. Using this result he shows that $E(n)=E(Z_n^2)/E(z^2)$ when $E(z)=0$ and then obtains a convenient approximation for $E(n)$ and upper and lower bounds for $E(n)$. Here Z_n is the sequential cumulative sum and $\varphi(t)$ is $E(e^{tz})$, where z is a term of Z_n; n is the number of terms in Z_n at the point of termination of the test.

A. M. Mood (Ames, Iowa).

Baillie, Donald C. On testing the significance of mortality ratios by the use of χ^2. Trans. Actuar. Soc. America 47, 326–344 (1946).

Hajós, G. Grundzüge der Fehlerabschätzung. Mat. Fiz. Lapok 49, 84–122 (1942). (Hungarian. German summary)

Nach Besprechung der Notwendigkeit, Vorteile und Prinzipien der Fehlerabschätzung wird das Ziel gesteckt, die wohlbekannten Elemente der Fehlerabschätzung systematisch darzustellen und sie an mancher Stelle zu ergänzen. Die Anwendungen der Wahrscheinlichkeitstheorie werden nicht gestreift. Im Sinne der Approximationsmathematik wird nur praktischen Zielen gedient.

From the author's summary.

Stihi, E. E. Sur la valeur réelle de la mesure d'une grandeur. Ann. Sci. Univ. Jassy. Sect. I. 26, 528–530 (1940).

Marcantoni, Alessandro. Pesi e correlazioni per misure indirette e condizionate. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) 6(75), 37–46 (1942).

TOPOLOGY

Birkhoff, G. D., and Lewis, D. C. Chromatic polynomials. Trans. Amer. Math. Soc. 60, 355–451 (1946).

A study of the function $P_n(\lambda)$, the number of ways of coloring P_n, a particular map on a sphere, in λ colors. It is a polynomial in λ of the nth degree, the chromatic polynomial of the map. Equalities in these polynomials are used in alternative proofs of many reductions for the four color problem first proved by methods involving Kempe chains and inequalities before 1922. The alternative proofs extend these results to the case of any number of colors. Several methods of breaking down $P_n(\lambda)$ in terms of specific configurations present in the map P_n such as m-gons or Kempe chains are given which lead to specific reduction formulas. These are used to calculate the coefficients for chromatic polynomials in a number of specific cases, and so build up experimental data for conjecturing new results. In particular, a relation of dominance which has the four color theorem as one consequence is proved for n less than 17, and conjectured to be always true. *P. Franklin.*

Egyed, L. Über die wohlgerichteten unendlichen Graphen. Mat. Fiz. Lapok 48, 505–509 (1941). (Hungarian. German summary)

The author proves the following theorem. Let there be given a graph G. A necessary and sufficient condition that we should be able to orient the edges of the graph so that any two vertices can be connected by a directed path is that every edge of G is contained in a cycle. Robbins had previously proved this for finite graphs [Amer. Math. Monthly 46, 281–283 (1939)]. *P. Erdös* (Syracuse, N. Y.).

Krausz, J. Démonstration nouvelle d'une théorème de Whitney sur les réseaux. Mat. Fiz. Lapok 50, 75–85 (1943). (Hungarian. French summary)

Let G_1 and G_2 be two graphs such that any two vertices are connected by at most one edge. Then if there exists a mapping of the edges of G_1 onto the edges of G_2 so that two intersecting edges correspond to two intersecting edges (and vice versa), G_1 and G_2 are isomorphic, except if G_1 is a triangle and G_2 a vertex with three edges emanating from it.

P. Erdös (Syracuse, N. Y.).

Szele, Tibor. Kombinatorische Untersuchungen über den gerichteten vollständigen Graphen. Mat. Fiz. Lapok 50, 223–256 (1943). (Hungarian. German summary)

Let there be given a directed complete graph. Rédei proved that the number of directed open paths which pass through every vertex is odd. The author gives a new proof of this result and proves several generalizations. He also investigates the maximum number of such paths.

P. Erdös (Syracuse, N. Y.).

Turán, Paul. Eine Extremalaufgabe aus der Graphentheorie. Mat. Fiz. Lapok 48, 436–452 (1941). (Hungarian. German summary)

Let there be given a graph G of n vertices. The author proves that if G has more than

$$\tfrac{1}{2}(k-2)(n^2-r^2)/(k-1)+\binom{r}{2}=d_k(n)$$

edges, then G contains a complete graph of order k. He also shows that there is a unique graph having exactly $d_k(n)$ edges and not containing a complete graph of order k. We can obtain this graph as follows: number the vertices by $1, 2, \cdots, n$. Two vertices are connected if and only if they are incongruent mod $(k-1)$. Some related problems are discussed. *P. Erdös* (Syracuse, N. Y.).

Tutte, W. T. A ring in graph theory. Proc. Cambridge Philos. Soc. 43, 26–40 (1947).

Let the complexity of a graph L be defined as the number of trees which can be formed by taking all the nodes and some (or all) of the branches; e.g., the vertices and edges of a tetrahedron form a graph of complexity 16. Let L_A' be derived from L by suppressing a given branch A, and L_A'' by identifying the two ends of A while suppressing A and any other branches that may have joined those ends. The complexity of L is equal to the sum of the complexities of L_A' and L_A''; e.g., $16=8+8$. The author seeks to characterize those numerical properties of a graph which are additive in this sense. He then considers "cubical" graphs whose nodes are all of degree 3, and describes a simple transformation by means of which any such graph may be reduced to a standard form consisting of the same (even) number of nodes joined in sequence by single and double branches alternately, with a loop at each end of the whole chain. *H. S. M. Coxeter* (Toronto, Ont.).

29 + mg math has 1-28 bet index 297

★**Tietze, H. Ein Kapitel Topologie. Zur Einführung in die Lehre von den verknoteten Linien.** Hamburger Math. Einzelschr. **36**, vii+47 pp. (1942).

This book (originally an hour-lecture at a meeting) contains an introduction to Minkowski's theory of quadratic forms with rational coefficients and to Reidemeister's theory of the quadratic form associated with a knot. No proofs are given. Several examples are carefully worked out; there are many figures. *H. Samelson* (Ann Arbor, Mich.).

Beda Neto, Luís. Contribution to the study of the theory of functions. VIII. Concept of function. Revista Fac. Ci. Univ. Coimbra **8**, 102–129 (1940); **9**, 69–95 (1941). (Portuguese)

This continues a detailed account of a general type of topological space. For the earlier parts, see the same Revista **4**, 3–48, 88–140 (1934); **5**, 129–171, 281–303 (1935); **6**, 290–326 (1937); 480–531 (1938). *R. P. Boas, Jr.*

Nachbin, Léopoldo. Sur la combinaison de topologies pseudo-métrisables et métrisables. C. R. Acad. Sci. Paris **223**, 938–940 (1946).

Let L be a complete lattice ("réticule achevé"). For any subset A of L let $S(A)$ denote the class of suprema sup x, $x \epsilon B$, of all nonvoid subsets of B; let $I(A)$ denote the dual object. The author considers the lattice L of topologies of a class E; he formulates the effect of applying some of the operations S, I, IS, SI, SIS, ISI, $ISIS$, $\cdots$, on M and N, the classes of metrizable and pseudometrizable topologies, respectively. In several cases familiar classes of topologies are obtained. *R. Arens* (Princeton, N. J.).

Doss, Raouf. Sur les espaces où la topologie peut être définie à l'aide d'un écart abstrait symétrique et régulier. C. R. Acad. Sci. Paris **223**, 1087–1088 (1946).

The symmetric regular écart (s. r. é.) was defined by Fréchet [Portugaliae Math. **5**, 121–131 (1946); these Rev. **8**, 48]. From his recent results [R. Doss, same C. R. **223**, 14–16 (1946); these Rev. **8**, 48] the author now concludes that a space can be given an s. r. é. if and only if it has an open base which is the union of a system of partitions which is linearly ordered by refinement. *R. Arens.*

Frenkel, Yanny. Criteria of bicompactness and of H-completeness in an accessible topological Fréchet-Riesz space. Ciencia y Técnica **107**, 383–401 (1946). (Spanish)

The author characterizes bicompact spaces in terms of the intersection properties of various types of families of sets. A typical one, in terms of open sets, follows. Let a family $\mathfrak{U}$ of open sets be called a T_1-system if (a) $V_1, \cdots, V_n \epsilon \mathfrak{U}$ implies $V_1 \cap \cdots \cap V_n$ is not void; (b) if an arbitrary set A fails to meet some $V \epsilon \mathfrak{U}$ then A has a neighborhood $W \supset A$ whose complement meets every $V \epsilon \mathfrak{U}$. Then a T_1-space X is bicompact if and only if for every maximal T_1-system which is also a dual ideal there is a point common to all its members. *R. Arens* (Princeton, N. J.).

Scorza Dragoni, G. Estensione alle quasi-traiettorie di un teorema di Brouwer sulle traiettorie di un automeomorfismo del piano. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **1**, 156–161 (1946).

Let t be an orientation-preserving fixed-point-free topological automorphism of the plane. A theorem of Brouwer asserts that simple arcs α and β with common end-points cannot form a simple closed curve if $\beta \cap t(\beta)=0$ and α is a trajectory arc which contains a translation arc, i.e., one which does not intersect its image and which has ends p, $t(p)$. The present paper contains an extension in which "translation arc" is replaced by "translation quasi-arc," which is a limit of translation arcs. *P. A. Smith.*

Scorza Dragoni, G. A proposito di un teorema sugli archi di traslazione di un autoomeomorfismo del piano, privo di punti uniti e conservante il senso delle rotazioni. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **1**, 697–704 (1946).

Let t be a fixed-point-free topological automorphism of the (x, y)-plane. Let λ be a translation arc of t, that is, a simple arc which does not intersect its image, and has end-points p, $t(p)$. Let Σ be one of the two regions into which the plane is separated by $\sigma(\lambda)=\cdots+t^{-1}(\lambda)+\lambda+t(\lambda)+\cdots$. Assume that λ is a finite polygonal line with sides parallel to the axes. Then either there exists a polygonal half-line (a finite polygonal line plus a ray) with sides parallel to the axes, which begins on $\sigma(\lambda)$ and otherwise lies in Σ and which fails to intersect its t-image; or there exists a finite polygonal line with sides parallel to the axes, which begins on $\sigma(\lambda)$ and otherwise lies in Σ, is a limit of translation arcs, and enjoys certain additional properties. *P. A. Smith* (New York, N. Y.).

Volpato, M. Un criterio per l'esistenza di elementi uniti nelle trasformazioni topologiche del cerchio. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **1**, 704–709 (1946).

Let G be a circular region bounded by g in the plane π. By a well-known theorem of Brouwer, a topological transformation of $G+g$ into π admits a fixed point if (1) $t(G) \subset G$. The author gives new sufficient conditions for a fixed point when (1) is not assumed. In addition to requiring that $t(G) \cap G \neq 0$, the conditions concern the disposition of the images of certain subarcs of $g+t(g)$ relative to certain regions bounded by that set. *P. A. Smith* (New York, N. Y.).

Hopf, Heinz. Bericht über einige neue Ergebnisse in der Topologie. Revista Mat. Hisp.-Amer. (4) **6**, 147–159 (1946).

This article [dated 1941] gives a sketch of some recent results in topoiogy, due primarily to Hopf and his students. The main subjects discussed are: vector fields on manifolds, with some algebraic applications; fibre spaces; topology of group manifolds; differential geometry in the large; relation of the fundamental group to the second homology group. *H. Whitney* (Cambridge, Mass.).

Kiang, Tsai-han. An application of the addition formulas of Mayer-Vietoris. Acad. Sinica Science Record **1**, 275–276 (1945).

Relations between the Betti numbers of a nonorientable 3-manifold M and its 2-leaved orientable covering are obtained for special M's. *N. E. Steenrod.*

Mayer, W. Singular chain intersection. Ann. of Math. (2) **47**, 767–778 (1946).

Continuing an earlier paper [same Ann. (2) **46**, 29–57 (1945); these Rev. **6**, 280], the author applies the theory of cup and cap products developed there to define the intersection of singular chains in a manifold and establish their properties. *N. E. Steenrod* (Ann Arbor, Mich.).

NUMERICAL AND GRAPHICAL METHODS

★**Fletcher, A., Miller, J. C. P., and Rosenhead, L. An Index of Mathematical Tables.** McGraw-Hill Book Company, New York; Scientific Computing Service Limited, London, 1946. viii+451 pp. $16.00, U. S. A.; 75/-, Great Britain.

This work, begun in 1939, is the result of a careful inspection, with a practical viewpoint, of the literature of mathematical tables and numerical analysis. In selecting material the authors have included about 2000 items and excluded some 1000. Tables having only historical interest have been omitted. Tables in the theory of numbers, algebraic tables such as symmetric functions, group theory tables, etc., statistical tables of special type, physical, geophysical and most astronomical tables were left out also. Those tables which were included are in most cases each described in a single line. Nevertheless the book occupies more than 350 pages, not counting its extensive bibliography. This gives some idea of the enormous task involved in its preparation and of its great service to mathematical analysis.

The book is in two parts. Part I, in 24 sections, is a description of the tables and is entitled "Index according to functions." A condensation of the classification of tables employed is as follows: §§ 1–5. Factor tables, powers, constants, Bernoulli numbers, etc. §§ 6–12. Logarithms, exponential, circular and hyperbolic functions, etc. § 13. Exponential, sine and cosine integrals. § 14. Gamma, beta and related functions. § 15. The error function and its relatives. § 16. Legendre functions. §§ 17–20. Bessel functions. § 21. Elliptic functions. § 22. Riemann zeta, hypergeometric, Mathieu functions, etc. § 23. Interpolation and related tables. § 24. Harmonic analysis.

Each section is subdivided into articles and each article is devoted, in most cases, to lists of tables of a single function. A decimal classification is used whose integer part is the section number and whose decimal part depends on the order of the type of function in a logical arrangement. This arrangement, as far as the first decimal place is concerned, is set forth in the introduction of each section. Thus one may find one's way quickly to any particular kind of table.

Under each article is given a one-line description of each table of the function concerned. Thus under article 13.15 ($\log_{10} \mathrm{Ei}(x)$) we find

6 dec. 4(.1)5(1)10, 20 Δ^2 Bellavitis 1874 (159).

This means that a 6 decimal table of $\log_{10} \mathrm{Ei}(x)$ for $x=4, 4.1, 4.2, \cdots, 5, 6, 7, \cdots, 10, 20$ with first and second differences is to be found on page 159 of Bellavitis 1874. This last is a reference to part II, the bibliography, arranged by authors, the date serving to identify the item among the author's list of works cited. Other symbols and devices, explained in the introduction, are used to convey other information in a concise manner. One of these is the use of boldface type to indicate a "standard" table, one which is outstanding and has a reputation for accuracy among a number of tables of the same function.

This compact presentation leaves no room for more than an infrequent line or two of comment about a particular table as to its accuracy or arrangement. No illustrations of the use of a table or extensive lists of errata are given. However, there is a quantity of very helpful and interesting comment on the functions themselves and on classes of tables. Much of this has to do with the explanation and correlation of the often vexing variances in notation used by table makers and other mathematicians. Section 4 on Bernoulli and Euler numbers and polynomials, etc., is particularly valuable as a reference, not only to tables, but also to notational matters. Future writers on these topics will do well to consult this section before adopting any new (or, for that matter, old) notation. Section 5, mathematical constants, etc., is also of value beyond its use as a reference to sources, since it gives the actual values of the constants considered often to more than 15 decimals. The reviewer has found this collection of constants extremely useful as a desk reference.

The bibliography, part II, occupies pages 373–444. Asterisks before the title indicate that the authors have not inspected the particular item; few asterisks appear. A glance at part II shows that a large number of items are part of papers published in periodical literature. This indicates the impracticability of building up a good library of mathematical tables without the help of periodicals. The reviewer believes that the usefulness of part II would have been increased had the authors attached to each item the number of that section (or sections) of part I involved in the table cited. This is especially true in cases where there are many items by the same author since distinguishing titles of papers are not given.

This work has been reviewed and errata noted in Math. Tables and Other Aids to Computation **2**, 13–18, 136, 178–181 (1946); 219–220, 277–278 (1947). The high percentage of minutiae among the errata and suggested addenda speaks well for the practical soundness of the book.

D. H. Lehmer (Berkeley, Calif.).

Cecconi, Arturo. Intersezioni di superficie cilindriche con formule e tabelle per le applicazioni tecniche. Ist. Veneto Sci. Lett. Arti. Parte II. Cl. Sci. Mat. Nat. **101**, 419–439 (1942).

Computation of the area of the intersection of two right circular cylinders. The integrals are evaluated numerically. There are tables to four decimal places and for the range $k=0(.01)1$ of the integrals

$$\tfrac{1}{2}\int_{-\pi/2}^{\pi/2}\{1-(h+k\sin\phi)^2\}^{\frac{1}{2}}d\phi,$$

$$\tfrac{1}{2}\int_{-\pi/2}^{\pi/2}\cos^2\phi\{1-(h+k\sin\phi)^2\}^{-\frac{1}{2}}d\phi,$$

either with $h=0$ or with $h=1-k$; in the former case the integrals are elliptic, in the second they can be expressed by elementary functions.

W. Feller (Ithaca, N. Y.).

Reynolds, Wm. A. A prepunched master deck for the computation of square roots on IBM electrical accounting equipment. Psychometrika **11**, 223–237 (1 plate) (1946).

Tortorici, P. Sulla risoluzione numerica delle equazioni in generale e in particolare di talune notevoli che si presentano in matematica finanziaria. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **1**, 582–586 (1946).

The author discusses a method for the approximate solution of equations. He applies this method to the problem of finding the rate of interest if the present value of a deferred or immediate annuity is given.

E. Lukacs.

Morris, J. An escalator process for the solution of linear simultaneous equations. Philos. Mag. (7) 37, 106–120 (1946).

The paper describes and illustrates a method which involves the introduction of auxiliary quantities which are obtained in a systematic manner by solving sets of equations in one unknown, two unknowns, three unknowns, etc. The author's summary states: "The process does not involve the usual devices of elimination or of the evaluation of determinants, neither does it involve iteration or the like." When presented in algebraic form the process does not resemble the usual method of elimination. Nevertheless, in the numerical calculation the successive numbers obtained in the escalator process are exactly the same as those found, for example, in Crout's method of elimination. In particular, the same divisions must be performed as in the case of elimination. This is important in view of the author's belief "that the escalator process is not unduly sensitive, even when the equations to be solved are ill-conditioned as regards the usual processes for solution." Since "ill-conditionedness" reveals itself by the occurrence of small divisors, with consequent magnification of errors, it is not apparent that the escalator process could be either better or worse than other methods of elimination. *W. E. Milne.*

Backman, Gaston. Rekursionsformeln zur Lösung der Normalgleichungen auf Grund der Krakovianenmethodik. Ark. Mat. Astr. Fys. 33A, no. 1, 14 pp. (1946).

Banachiewicz [Astr. J. 50, 38–41 (1942); these Rev. 4, 90] introduced "Cracovians" (which are really matrices, except for interchange of rows and columns in the rule for multiplication) for the solution of simultaneous linear equations. The present author, recognizing the unfamiliarity with Cracovians on the part of research workers in applied fields, has undertaken to present the steps of the solution of equations by means of recursion formulas so as to avoid the formal use of Cracovians (or matrices).

W. E. Milne (Corvallis, Ore.).

Mazzarella, Franco. Semplificazione alla soluzione di un sistema di equazioni lineari. Rend. Accad. Sci. Fis. Mat. Napoli (4) 13, 197–201 (1945).

Berry, Clifford E., and Pemberton, J. C. A twelve-equation computing instrument. Instruments 19, 396–398 (1946).

Cf. Berry, Wilcox, Rock, and Washburn, J. Appl. Phys. 17, 262–272 (1946); these Rev. 7, 488.

Vernotte, Pierre. À propos de la représentation d'une loi expérimentale par une loi approchée et une courbe d'écart. C. R. Acad. Sci. Paris 223, 1105–1107 (1946).

Rossier, Paul. Sur la courbe d'erreur relative au tracé de la tangente en un point d'une courbe graphique. C. R. Séances Soc. Phys. Hist. Nat. Genève 63, 85–86 (1946).

Fulcher, Gordon S. Interpolation with the aid of a plot of first differences. J. Appl. Phys. 17, 617–628 (1946).

The author develops a semi-graphical method of interpolation employing a plot of first differences, which is essentially equivalent to interpolating by the classical finite difference formulae, with an error of about one-sixth of the fourth difference term in the Newton-Bessel formula.

T. N. E. Greville (Washington, D. C.).

Strachey, C., and Wallis, P. J. An expression for the sine of a Fourier series. Philos. Mag. (7) 37, 84–86 (1946).

A numerical table is given which simplifies the calculation of the coefficients K_m in the expansion

$$\exp\left\{i\left(A\theta+B+\sum_{r=1}^{n}C_r\sin(r\theta+\delta_r)\right)\right\}=\sum_{-\infty}^{\infty}K_m\exp(im\theta).$$

I. Opatowski (Ann Arbor, Mich.).

Tolstov, Yu. G. A new electrical apparatus for harmonic analysis and synthesis. Bull. Acad. Sci. URSS. Cl. Sci. Tech. [Izvestia Akad. Nauk SSSR] 1946, 389–400 (1946). (Russian)

The apparatus described is an analogy device composed of electrical resistances using ordinary alternating current. The heart of it is a set of "multipliers." Each multiplier consists of a potentiometer (with movable tap) and a voltage divider (with fixed taps for taking off fractions proportional to the sines of evenly spaced submultiples of 90°). By means of the potentiometer, an arbitrary fraction of the maximum voltage can be applied to the outside terminals of the divider; this fraction can be reversed in sign by a reversing switch. When the machine is used for analysis, a multiplier is assigned to each ordinate, and the value of the ordinate is then set on the scale of the potentiometer; when it is used for synthesis (i.e., summation to find the ordinates when the coefficients are given), a multiplier is similarly assigned to each coefficient. The multipliers are wholly isolated from one another, since the applied voltage for each is taken from the secondary of a separate transformer. If, now, appropriate voltages from the various dividers are connected in series, the total voltage across the connection will be proportional to a coefficient or ordinate being sought. A separate connection has to be made in this way for each quantity to be found; but the connections are independent of the data, and consequently they can be programmed beforehand on the controls of the apparatus. The device described is arranged for 36 ordinates and harmonics up to the 18th. The accuracy claimed is about 1 per cent of maximum. *H. B. Curry* (State College, Pa.).

Fürth, R., and Pringle, R. W. A photo-electric Fourier transformer. Philos. Mag. (7) 37, 1–13 (1946).

An instrument, based on photo-electric measurements, is described in which the Fourier transform of a given function is produced in the form of a trace on the screen of a cathode-ray oscilloscope. The function to be transformed is introduced either as a mask cut out in the shape of a graph or as a film in which the variation of light transmission represents the given function. Specifically, the instrument evaluates the integral

$$\psi(y)=\int_a^b f(x)\cos(yx+\delta)dx.$$

It is a limitation of the design shown that $f(x)$ must be positive within the interval $a\leqq x\leqq b$. A constant correction term may be introduced in order to satisfy this condition, but it is then necessary to correct the output of the machine. [Another solution of the same problem, i.e., the problem of producing negative light, is described by Hazen and Brown, J. Franklin Inst. 230, 19–44, 183–205 (1940); these Rev. 2, 62.] The above limitation does not detract from the usefulness of the machine in many practical problems.

The construction of the machine is given with sufficient detail to guide a new design. There is some question whether the location of the axis of a cylindrical lens in the optical

system is correctly described. A number of examples of the machine's performance are shown and calculated curves for the same cases are displayed for comparison. These comparisons indicate that the instrument performs exceptionally well. *S. H. Caldwell* (Cambridge, Mass.).

Akushsky, I. J. On certain schemes of the numerical harmonical analysis. C. R. (Doklady) Acad. Sci. URSS (N.S.) **52**, 471–474 (1946).

The author describes two rapid schemes for computing approximate Fourier coefficients by means of punched cards. *E. Bodewig* (The Hague).

Akushsky, I. Numerical solution of the Dirichlet equation with the aid of perforated card machines. C. R. (Doklady) Acad. Sci. URSS (N.S.) **52**, 375–378 (1946).

The author gives directions for two procedures for employing standard punched card machines to solve Dirichlet's problem in a rectangular region, by the method of iteration of Liebmann's formula. Only punching, sorting, and tabulating machines are used. Maximum eigenvalues of the equation $\Delta f - \lambda f = 0$ can also be obtained. In one of the procedures, three sets of cards are punched, a card in each set for each point of the network. One set bears, in addition to indication data, the trial values of the function, another set bears the boundary values and the third set has indication data only. The indication data are supplied by punching numerical indices which are associated with each point of the net. The cards are ordered by the sorting machine according to a certain column and then summed with automatic control on another column. The results when printed by the tabulator along with the same indication data give all the information necessary for repunching a new set of cards for a repetition of the procedure, except that the new trial values are multiplied by 4. *P. W. Ketchum.*

Bruk, I. S. A mechanical device for the approximate solution of the Poisson-Laplace equations. C. R. (Doklady) Acad. Sci. URSS (N.S.) **53**, 311–312 (1946).

An array of racks and gears is arranged so that the displacement of any central rack is proportional to the arithmetic mean of the displacements of the four adjacent racks which occupy the corners of a square with the central rack located at the intersection of its diagonals. By this means, the difference equation derived from the Poisson or Laplace equation may be evaluated. Auxiliary slides are provided in the racks for introducing the function given in Poisson's equation or for setting this function everywhere equal to zero in Laplace's equation. *S. H. Caldwell.*

Bruk, I. S. A device for the solution of ordinary differential equations. C. R. (Doklady) Acad. Sci. URSS (N.S.) **53**, 523–526 (1946).

This is a preliminary announcement of a device which is to be described in further detail later. An electronic procedure is described for obtaining solutions of ordinary differential equations by a method similar to that used in the differential analyzer. The independent variable is a sinusoidal function of time and is used to control the horizontal sweep of a cathode-ray oscilloscope. Although the basic integrating circuit is limited to integration with respect to time, it is possible to integrate with respect to any other variable by writing the integral in the form $\int y d\varphi = \int y(d\varphi/dt)dt$. The necessary multiplication is performed in a balanced modulator. Initial conditions are introduced by means of step functions in synchronism with the independent variable. The output voltage which represents the dependent variable of the equation is used to control the vertical motion of the cathode-ray oscilloscope, and the combination of this deflection and the horizontal deflection produced by the independent variable produces a stationary pattern on the oscilloscope which is the desired solution for the particular initial conditions used. Functions are introduced by producing them as the solutions of auxiliary differential equations by methods similar to those used with a differential analyzer.

While the device described is of relatively low accuracy, it produces solutions very rapidly. There are many problems in which its precision is adequate, and in other problems where better final results are desired the device can be used for rapid exploration in order to determine regions of interest. *S. H. Caldwell* (Cambridge, Mass.).

Amble, O. On a principle of connexion for Bush integrators. J. Sci. Instruments **23**, 284–287 (1946).

The integrators used in differential analyzers are so constructed that the motions of their shafts are related by the equation $dW = kUdV$. Ordinarily this equation is used in its integrated form $W = k\int UdV$, and because of mechanical limitations the process is irreversible. A method is described which uses an adding gear in combination with the integrator so that the combination performs the operation $V = k^{-1}\int U^{-1}dW$. With this type of connection greater economy in the use of differential analyzer components is frequently possible.

The author calls this a "regenerative connection." The same result is obtained without using an adding gear when servomechanisms are used to drive the integrator shafts. [Bush and Caldwell describe the latter system, called "inverse integration," in J. Franklin Inst. **240**, 255–326 (1945), in particular, p. 316; these Rev. **7**, 339.] *S. H. Caldwell* (Cambridge, Mass.).

Sauer, R., und Pösch, H. Zur Theorie der Integriermaschine für gewöhnliche Differentialgleichungen. Z. Angew. Math. Mech. **24**, 63–70 (1944). [MF 13191]

The theory of the differential analyzer is discussed and the class of differential equations solvable by an analyzer is determined. Two diagrams are employed for this purpose: the ordinary connection diagram and the force-flow diagram. The latter insures that the integrators have their input and output sides properly oriented. The force-flow diagram also forms a means of classifying the connection diagrams into series or parallel, reducible or irreducible, etc. It is shown that any differential equation of the form $P(x, y, y', \cdots, y^n) = 0$, where P is a polynomial, can be solved automatically by a differential analyzer with constant inputs. The same result in somewhat more general form was proved earlier by Shannon [J. Math. Phys. Mass. Inst. Tech. **20**, 337–354 (1941); these Rev. **3**, 279]. *P. W. Ketchum* (Urbana, Ill.).

Lemaître, G. Intégration d'une équation différentielle par itération rationnelle. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **28**, 815–825 (1942). [MF 13680]

The author has previously [same Bull. (5) **28**, 347–354 (1942); these Rev. **7**, 218] described a method (called rational iteration) designed to accelerate the orthodox process of iteration. In this paper he applies the method to solving the differential equation (1) $dy/dx = 2y^2(y-x)$ with initial conditions $y \sim x$ as $x \to \infty$. Having found an initial approximation $y_0(x)$ to the solution of (1) by expanding

y_0-x in powers of $1/x$, the author first uses orthodox iteration and defines a sequence of functions $y_n(x)$ by the recurrence (2) $y_{n+1}(x)=2\int_x^\infty(y_n-u)du$, which, if convergent, tends to the solution of (1). Rational iteration now consists in breaking off the recurrence (2) at $n=2$, defining y_3 by

$$(3)\qquad y_{n+3}=(y_n+y_{n+2}-y_{n+1}^2)/(y_n-2y_{n+1}+y_{n+2})$$

(with $n=0$), then using (2) to produce y_4 and y_5 and reverting to (3) (with $n=3$) to find y_6 from y_4 and y_5, and so on.

Although this is not stated in this paper, the formula (3) can be derived by assuming that the $y_{n+1}-y_n$ are in geometric progression with a constant ratio q (say) given by $q=(y_{n+2}-y_{n+1})/(y_{n+1}-y_n)$ and the difference $y_{n+3}-y_{n+2}$ in (3) is in fact equal to a geometric series with the constant ratio q and an initial term given by $q(y_{n+2}-y_{n+1})$. This device of "forecasting" the course of iteration as a geometric progression is well known to those practising iteration. The process appears to be successful in the above example.

H. O. Hartley (London).

Schmidt, Ernst. Das Differenzenverfahren zur Lösung von Differentialgleichungen der nichtstationären Wärmeleitung, Diffusion und Impulsausbreitung. Forschung Gebiete Ingenieurwesens. Ausg. B. 13, 177–185 (1942).

The paper presents a graphical procedure for the solution of Fourier's heat equation for plates, cylinders, and spheres. The partial differential equation is first replaced by a partial difference equation with the time and space intervals so chosen that the temperature at time $t+\Delta t$ is obtained by a simple graphical construction from the known temperatures at time t. For the case of the plate the space intervals are equal. For the cylinder and sphere the space interval is a function of the radius vector at the point. By a modification of the procedure the solution is adaptable to the case of variable conductivity and heat capacity, and to the case where the boundary conditions vary with time.

W. E. Milne (Corvallis, Ore.).

Herget, Paul. Numerical integration with punched cards. Astr. J. 52, 115–117 (1946).

This paper describes a method for numerical integration, with punched cards, of the equations of motion of the three- or n-body problem. The method is one of successive approximations and consists essentially of building up, over short arcs, a table of coordinates from the second differences (under the assumption that fourth and higher differences are negligible). A knowledge of preliminary values of the coordinates is therefore prerequisite. The use of punched cards for such integrations is not new [cf., for instance, Eckert, Astr. J. 44, 177–182 (1935)], but the author claims to have reduced it to a team-work of standard types of machines such as a tabulator and a multiplying punch, and to have achieved so rapid a convergence that all coordinates which differ by about 100 units of the last place or less from their values on the preceding cycle would remain unchanged by a recomputation. The average time required for the completion of one cycle is 1.0 min. per single interval of integration of the problem of three bodies on the 405 tabulator, and 1.2 min. per step on the 601 multiplying punch. Since each machine is idle when the other is in operation, the overall efficiency can be increased if two (or more) arcs are being computed concurrently.

Z. Kopal.

Motz, H. The treatment of singularities of partial differential equations by relaxation methods. Quart. Appl. Math. 4, 371–377 (1947).

Ballarin, Silvio. Grafici e formole per l'interpretazione di rilievi gravimetrici eseguiti a scopo di prospezione. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) 4(73), 497–524 (1940).

Four different graphical methods with corresponding tables are developed in this paper to facilitate the computation of the gravity acceleration Dg caused at a given point by a cylindrical distribution of excess-masses of constant density-contrast, the axis of the cylindrical body or tectonic structure (the cause of the observed axial anomaly) being horizontal and of any known normal section. Only the direct problem is solved, the location of the cylinder and the shape and dimensions of its normal section being considered as known. The methods proposed are not new, but the graphs and tables have not been published before.

E. Kogbetliantz (New York, N. Y.).

Volta, Luigi. Considerazioni intorno ad una formola del somigliana sulla gravità terrestre. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) 8(77), 469–477 (1944).

Boaga, Giovanni. Le anomalie gravimetriche e le deviazioni della verticale per pianeti sferoidici non di rotazione. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) 5(74), 455–468 (1941).

Boaga, G. Sulla compensazione rigorosa per direzioni delle catene geodetiche. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 1, 199–205 (1946).

Legros, Roger. L'échelle hyperbolique, généralisation des échelles linéaire et logarithmique. Ann. Physique (12) 1, 335–356 (1946).

The author discusses the graduation of a scale by the function $u=A+B\log\sinh^{-1}\frac{1}{2}Cx$. It is shown that the use of such a scale (called a hyperbolic scale) may sometimes improve the legibility of a diagram.

E. Lukacs.

Baudouin, Georges. Principe d'une règle à calcul présentant une échelle logarithmique de grande longueur. C. R. Acad. Sci. Paris 224, 96–97 (1947).

A slide rule with a very long logarithmic scale is constructed. This scale is drawn on a sine curve and broken into ten parts to make the rule handy. A reading device consisting of two concentric circles is used.

E. Lukacs.

Polidori, C. Il problema dei capitali accumulati. Univ. Roma e Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) 3, 203–215 (1942).

Let $f(x, y)$ be the amount at time y $(y>x)$ of a monetary unit invested at time x. Here the moment of investment x is taken as the beginning of the process of capitalization. However, if the instant a is taken as the moment of reference the amount at time y of a monetary unit invested at time x is given by ${}_af(x, y)=f(a, y)/f(a, x)$ $(a<x<y)$. Starting from a law of capitalization $f(x, y)$ the author considers the operation of a bank which makes payments and collects premiums. He discusses the following four types of financial operations. (1) The payments of the bank and the premiums due to the bank are both evaluated by the same function ${}_af(x, y)$. (2) The payments of the bank and the premiums due to the bank are both evaluated by a function $f(x, y)$. (3) The premiums are evaluated by the function ${}_af(x, y)$, the payments of the bank by the function $f(x, y)$. (4) The premiums are evaluated by the function $f(x, y)$, the payments of the bank by ${}_af(x, y)$. Here a denotes the beginning

of the process of capitalization, x the moment at which a payment or a premium is due, y the moment of valuation.
E. Lukacs (Cincinnati, Ohio).

Féraud, Lucien. Sur les formules de l'assurance invalidité. Mitt. Verein. Schweiz. Versich.-Math. **46**, 237–244 (1946).

It is a well-known fact that the probabilities q_x, q_x^{aa}, q_x^i, i_x used in invalidity insurance are connected by an equation. The author remarks that certain formulae used for those probabilities are inconsistent with this equation. This is due to the fact that the formulae referred to are only approximations.
E. Lukacs (Cincinnati, Ohio).

Ludwig, G. Eine Methode zur approximativen Berechnung der Werte temporärer Leibrenten. Mitt. Verein. Schweiz. Versich.-Math. **46**, 215–230 (1946).

Ammeter, Hans. Das Maximum des Selbstbehaltes in der Lebensversicherung unter Berücksichtigung der Rückversicherungskosten. Mitt. Verein. Schweiz. Versich.-Math. **46**, 187–213 (1946).

Hagstroem, K. G. La riserva prospettiva dell'assicurazione generale sulla vita e la misura del contenuto assicurativo del contratto. Giorn. Ist. Ital. Attuari **12**, 103–121 (1941).

After some remarks on retrospective calculation of policy values the author discusses the possibility of finding a rational measure of the amount of insurance of a given policy.
P. Johansen (Copenhagen).

de Finetti, B. Impostazione individuale e impostazione collettiva del problema della riassicurazione. Giorn. Ist. Ital. Attuari **13**, 28–53 (1942).

Wenn zwei Versicherungsgesellschaften A und B gegenseitig einen Teil ihrer Versicherungsbestände rückversichern, so gibt es nach dem Verfasser gewisse Verteilungen (die sich in einem geometrischen, zweidimensionalem Schema als Punkte einer Linie darstellen) bei denen sich ein "Optimum" einstellt; d.h. in geometrischer Deutung, es gibt Punkte, durch deren Variation man die Lage der Gesellschaft A nicht verbessern kann, ohne gleichzeitig die von B zu verschlechtern, und umgekehrt. Bei einer gegenseitigen Quotenrückversicherung ergibt eine proportionelle Verteilung ("equipartizione") beider Versicherungsbestände unter A und B ein solches Optimum. Die Betrachtungen werden auf beliebig viele Gesellschaften und auch auf die andern üblichen Arten der Rückversicherung ausgedehnt und ferner an die erhaltenen mathematischen Ergebnisse allgemeine wirtschaftliche Bemerkungen angeknüpft, welche den Vorteil koordinierter (gemeinwirtschaftlicher) Entscheidungen gegenüber einem System, im dem jeder nur mit Rücksicht auf das eigene Interesse die Entscheidung fällt, beweisen sollen.
P. Thullen (Quito).

Lordi, L. Sulle tavole di mortalità che portano alle stesse riserve matematiche. Giorn. Ist. Ital. Attuari **13**, 57–65 (1942).

When the force of interest and the force of mortality μ_x are given for a whole life insurance or an endowment, then one may find another force of mortality $\mu_x{}^I$ which gives exactly the same policy values. However, one cannot find a force of mortality $\mu_x{}^I$ which gives the same two sets of policy values for two different forces of interest.
P. Johansen (Copenhagen).

Del Chiaro, A. Sulle tavole di mortalità. Giorn. Ist. Ital. Attuari **12**, 81–102 (1941).

A general formula for determining the rate of mortality given by the author in an earlier paper [same Giorn. **11**, 214–232 (1940); these Rev. **7**, 465] is compared with similar formulae used in various countries at different times.
P. Johansen (Copenhagen).

Ottaviani, G. Sulle tavole di mortalità. Giorn. Ist. Ital. Attuari **13**, 66–76 (1942).

The author compares different formulae for determining the rate of mortality.
P. Johansen (Copenhagen).

Sibirani, Filippo. Sopra le funzioni di sopravvivenza. Univ. Roma e Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **3**, 79–89 (1942).

Denote by l_x the number living in a mortality table and by ${}_tp_x = l_{x+t}/l_x$ the probability that a person of age x will survive t years. The author proves the following three theorems. (a) If any two ages x and y determine an age z such that ${}_tp_x \cdot {}_tp_y = {}_tp_z$ for any value of t then the mortality table follows Gompertz' law, i.e. $l_x = kg^{c^x}$. (b) If any two ages x and y determine an age z such that ${}_tp_x \cdot {}_tp_y = {}_tp_z{}^2$ for any value of t then the mortality table either follows Gompertz' law or Makeham's law ($l_x = ks^x g^{c^x}$) or has one of the forms $l_x = kg^{x^2}$, $l_x = ks^x g^{x^2}$. (c) If for any value of t the probability ${}_tp_x = \varphi(t)[p_x]^{\psi(t)}$ (where $p_x = l_{x+1}/l_x$), then l_x has one of the forms $l_x = ks^x g^{c^x}$ or $l_x = ks^x g^{x^2}$ or $l_x = kg^{x^2}$.

Theorems (a) and (b) are special cases of Quiquet's theorem [Bull. Trimest. Inst. Actuaires Français **4**, 97–186 (1893)]. The proofs given by the author are of a more elementary nature than Quiquet's general discussion and have therefore some methodological interest. Another investigation related to Quiquet's theorem was made by Lubin [Festskrift til J. F. Steffensen, Copenhagen, 1943, pp. 100–108; these Rev. **8**, 58].
E. Lukacs (Cincinnati, Ohio).

ASTRONOMY

Garwick, Jan V. A new method of calculating general perturbations of asteroids and comets. Astrophys. Norvegica **3**, 281–299 (1943).

The distinguishing feature of this method is that the disturbed eccentric anomaly is used as independent variable. The coordinates used are cylindrical coordinates ρ, v, z. The plane $z=0$ is chosen to coincide very nearly with the orbital plane of the perturbed body; hence the difference between the radius vector r and ρ is of the order z^2, and has the square of the disturbing mass as a factor.

In undisturbed motion the radius vector satisfies the differential equation $d^2r/dE^2 + r - a = 0$. In disturbed motion the equation becomes $d^2\rho/dE^2 + \rho - a = P$, in which P has the disturbing mass as a factor. This equation has a suitable form for integration by successive approximations. An advantage of this method is that P contains in addition to contributions that depend on the disturbing function and its derivatives only terms having z^2 and $(dz/dE)^2$ as factors. The equations for the remaining variables are similar to those that arise in related methods.
D. Brouwer.

Opalski, W. Sur le mouvement du centre de gravité du système de deux corps rayonnants. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 1, 840–846 (1946).

The aim of this paper is to establish necessary and sufficient conditions under which the motion through space of the center of gravity of a two-body system, the components of which are constantly losing mass by radiation, will remain rectilinear and uniform. The author finds that this will be the case provided that the ratio of both masses remains constant in time, which is physically plausible only if the initial masses of both bodies are equal. In the majority of known binary stars this latter condition is not fulfilled, and therefore the motions of their centers of mass should, strictly speaking, depart from uniformity. An actual evaluation of the magnitude of such departures for eleven well-known close eclipsing systems discloses, however, that the ratio of the acceleration of the translational motion, arising from unequal loss of mass of both components by radiation, to the orbital acceleration is a quantity of the order of 10^{-11}, and therefore far too small to produce any observable effects.
Z. Kopal (Cambridge, Mass.).

Belorizky, David. Sur deux cas particuliers du problème des trois corps. C. R. Acad. Sci. Paris **223**, 193–196 (1946).

The author shows that the equations of the restricted three body problem admit of exact integration in the following two cases: first, when the two finite bodies have equal mass and describe parabolas in the (x, y)-plane about their center of mass, while the infinitesimal body moves (with suitable initial conditions) on the z-axis; second, when the parabolas just mentioned degenerate into portions of the x-axis, while the infinitesimal body moves in the (y, z)-plane; in both cases terms in the differential equations containing powers of z/r or y/r higher than the second are neglected, r being the distance of either finite mass from the origin, taken at the center of mass. *W. Kaplan.*

Dramba, Constantin. Les chocs triples imaginaires dans le problème général des trois corps. Mathematica, Timişoara 22, 74–80 (1946).

The author indicates briefly the form of the solutions of the three-body problem im the neighborhood of an imaginary triple collision. [Cf. Sémirot, C. R. Acad. Sci. Paris **212**, 974–977 (1941); these Rev. **5**, 79.] The variables are expressed as series in powers of $(t-t_0)^a$, a being a complex number depending on ten arbitrary constants.
W. Kaplan (Ann Arbor, Mich.).

Gratton, Livio. Sopra alcune proprietà dinamiche dei sistemi stellari. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) 2, 1–12 (1940).

This paper essentially casts the basic equations in Chandrasekhar's theory of stellar dynamics [Astrophys. J. **90**, 1–154 (1939); these Rev. **1**, 60] into more symmetrical form by introducing conjugate momenta instead of the velocities used by Chandrasekhar. Furthermore, the equations are written down for a general nonorthogonal system of coordinates and the more elementary results of the theory are also derived. *S. Chandrasekhar* (Williams Bay, Wis.).

Garwick, Jan V. Note on stellar systems with ellipsoidal velocity-distribution. Astrophys. Norvegica 3, 301–305 (1943).

This paper is concerned with the general case of a stellar system in which there exists an ellipsoidal velocity-distribution. If, in such a system, we represent the number of stars with rectangular coordinates x to $x+dx$, y to $y+dy$, z to $z+dz$, and with velocity components u to $u+du$, v to $v+dv$, w to $w+dw$, by $\Psi dudvdwdxdydz$, then we make the assumption that the distribution function can be written as $\Psi(Q, x, y, z, t)$, where $Q(u, v, w, x, y, z, t)$ represents an ellipsoid for constant x, y, z, t, but variable u, v and w. Schwarzschild originally wrote $\Psi=\varphi(x, y, z, t)e^{-Q}$. Chandrasekhar generalized this to read $\Psi=\Psi\{Q+\sigma(x, y, z, t)\}$. It is shown in the present paper that this is the most general form for the distribution function. *B. J. Bok.*

Jehle, Herbert. Statistical hypotheses in stellar dynamics. Physical Rev. (2) **70**, 538–555 (1946).

This paper develops a statistical theory of stellar systems, introducing a new assumption analogous to the quantum mechanical uncertainty principle. The assumption is that statistically independent elements of a stellar system cannot be crowded into a Boltzmann's phase space more closely than corresponding to an expectation value of one element in a given elementary volume. From this is deduced an exclusion principle, referring to a minimum distance between stars in position space. The classical hydrodynamical equations are expressed in terms of an expectation function and transformed to a form similar to the Schrödinger equation. *G. Randers* (Oslo).

Arrighi, Gino. Considerazioni sui moti lenti dei mezzi continui disgregati. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) 4, 473–476 (1943).

In this paper an error in sign in a memoir of Levi-Civita [Scritti Mat. Off. a L. Berzolari, Pavia, 1936, pp. 161–168] is corrected. [Cf. García, Revista Ci., Lima **45**, 463–483 (1943); these Rev. **5**, 191, where the same error is corrected.]
W. Kaplan (Ann Arbor, Mich.).

Agostinelli, Cataldo. Sulla variazione delle velocità angolare terrestre durante una lunazione. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) 8(77), 527–547 (1944).

We study the phenomenon of the periodic variation of the terrestrial angular velocity during a lunation, caused by the variability of the mass and the configuration of the tides due to the lunisolar attraction, taking account also of the internal viscosity of the tidal fluid layer.
From the author's summary.

Agostinelli, Cataldo. Equilibrio relativo di uno strato liquido omogeneo su di una sfera rigida rotante uniformemente. Boll. Un. Mat. Ital. (2) 5, 227–233 (1943).

Mendes, Marcel. La rotation de l'ellipsoïde hétérogène étudiée au moyen des fonctions de Lamé. J. Math. Pures Appl. (9) **24**, 51–72 (1945). [MF 15968]

Consider a fluid composed of n homogeneous ellipsoidal layers all rotating around the same axis. The density of the layers increases towards the center. It is shown that, if the fluid is in equilibrium under the effect of gravitational and centrifugal force, then the exterior ellipsoid must be an ellipsoid of Maclaurin or Jacobi. If all the points of one ellipsoid have the same angular velocity, then all the ellipsoids are confocal. If all the ellipsoids are confocal, their common axis of rotation is the minor axis. From this follows a theorem due to Hamy: a fluid mass in relative equilibrium in which the density increases constantly from the surface to the center cannot be composed of homogeneous

ellipsoidal layers. The paper also shows that, if a certain determinant is zero, new equilibrium figures can be found close to the confocal ellipsoidal solutions.
B. Friedman (New York, N. Y.).

Ruiz Wilches, Belisario. Study of a figure of equilibrium. Univ. Nac. Colombia **4**, 275–283 (1945). (Spanish) [MF 14663]

The author assumes that the centrifugal force is given by twice its usual value: $2mv^2/r$. He also makes the tacit assumption that the gravitational force on the earth's surface is equal to that produced if the earth's mass were concentrated at the center. From these assumptions a sixth degree equation for the shape of the earth is derived. The author claims this equation agrees closely with the result of geodetic measurements. [See the two following reviews.]
B. Friedman (New York, N. Y.).

Rozo M., Dario. Justification of the hypothesis of Ruiz Wilches. Univ. Nac. Colombia **6**, 359–362 (1946). (Spanish)

[See the preceding review.] The author tries to justify the assumption made in the preceding paper that the centrifugal force should be $2mv^2/r$. The argument is not convincing. *B. Friedman* (New York, N. Y.).

Carrizosa Valenzuela, Julio. Critique of the study of a possible equilibrium figure of the terrestrial globe. Revista Acad. Colombiana Ci. Exact. Fis. Nat. **6**, 459–466 = Univ. Nac. Colombia **5**, 341–361 (1946). (Spanish)

[See the two preceding reviews.] A critical examination is made of the two previous papers. The errors are pointed out and the claimed agreement with geodetic measurements is disputed. *B. Friedman* (New York, N. Y.).

Ledoux, P. On the dynamical stability of stars. Astrophys. J. **104**, 333–346 (1946).

The principal radial oscillations are considered for stars homogeneous in density and stars constructed on the standard model, when the adiabatic index Γ is nonuniform. The periods are determined partly by an approximation assuming a uniform dilation of the star, partly by using series solutions of the equations of motion. The condition of stability is found (a) when $\Gamma=\frac{5}{3}$ in a central core, $\Gamma<\frac{4}{3}$ outside, (b) when $\Gamma=1$ in the core, $\Gamma=\frac{5}{3}$ outside, (c) when $\Gamma=1$ in a shell between two spheres concentric with the star, $\Gamma=\frac{5}{3}$ outside. The region where $\Gamma<\frac{4}{3}$ must be extensive for instability; for example, in (a), when $\Gamma=1$ in the outer region, this must extend up to a point where the temperature is about half that at the centre.
T. G. Cowling (Bangor).

MECHANICS

Locatelli, P. Principi della statica delle costruzioni nella dinamica. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) **4(73)**, 157–167 (1940).

Poncin, Henri. Sur l'équilibre d'un système matériel illimité dans une direction donnée et sollicité par des actions normales à cette direction. C. R. Acad. Sci. Paris **223**, 1093–1094 (1946).

Artobolevsky, I. I. On cam mechanisms equivalent to slider-crank mechanisms. C. R. (Doklady) Acad. Sci. URSS (N.S.) **52**, 749–752 (1946).

It is shown that the relation between the linear motion of a slide and the rotational motion of a crank which are joined by a connecting rod can be simulated by a follower acting against an elliptical, circular or hyperbolic cam depending on whether the connecting rod is, respectively, longer than, equal to or shorter than the crank arm. *M. Goldberg.*

Rauter, Herbert. Die Chaslessche Elementargeometrie der Bewegung im Raume. Deutsche Math. **7**, 383–405 (1944).

Many of Chasles' classical theorems on the kinematics of a rigid body (or congruent transformations in space) were stated without proof [C. R. Acad. Sci. Paris **52**, 77–85, 189–197, 487–501 (1861)]. The most famous is theorem 66: every rigid motion may be regarded as a screw-displacement, which Chasles had discovered in 1830. As the author admits, proofs were soon supplied, e.g. by Brisse [J. Math. Pures Appl. (2) **19**, 221–264 (1874); (3) **1**, 141–180 (1875)]. Nevertheless, he proves them again, in what is perhaps a more straightforward manner. *H. S. M. Coxeter.*

Udeschini, Paolo. Sulla composizione delle forze con la regola del parallelogramma. Period. Mat. (4) **24**, 84–99 (1946).

Udeschini, Paolo. Movimento di continui i cui elementi differiscono soltanto per condizioni iniziali. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) **8(77)**, 503–514 (1944).

Cisotti, Umberto. Momenti d'inerzia di configurazione e loro intervento nella dinamica dei sistemi materiali. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **1**, 656–659 (1940).

Caldonazzo, Bruto. Considerazioni elementari sulla composizione di spostamenti rigidi. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) **7(76)**, 121–126 (1943).

Sestini, Giorgio. Sulla composizione dei moti rigidi. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) **7(76)**, 144–150 (1943).

Cattaneo, Carlo. Alcuni teoremi di minimo in dinamica e in cinetostatica. Univ. Roma e Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **2**, 321–335 (1941).

The paper contains an extension of the principles of least constraint and of least curvature of Gauss and Hertz. The author generalizes the Gaussian constraint by a concept of relative constraint [reviewer's terminology], which he obtains by suppressing only some of the kinematic constraints of the system and not necessarily all of them, as Gauss did. He defines the measure of the relative constraint by $G=\sum_i m_i\overline{Q_iP_i}^2$, where P_i is the actual position of the mass m_i at a time t and Q_i is the position that m_i would have at t if some of the kinematical constraints of the system were suppressed. He proves that the motion of a system subject to kinematical constraints has the smallest G among all motions which are possible when any number of the constraints are suppressed, the initial conditions being the same for all motions and the durations of the motions being sufficiently small. An addition theorem for G's corresponding

to different kinematical constraints is proved. For motions by inertia, the author uses the concept of the trajectory of the system which is defined as in the Hertz principle. He generalizes this principle by comparing the geodetic curvature of the actual trajectory with the geodetic curvatures of those trajectories that would be obtained if any number of the existing kinematical constraints were suppressed. Finally he defines a function which acquires a minimum value when all the reactions compatible with the constraints become the actual reactions. *I. Opatowski.*

Kochin, N. E. Release of dynamic systems from constraints. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 10, 541–544 (1946). (Russian. English summary)

This note deals with the properties of the Gaussian constraint. The same subject has been extensively studied by C. Cattaneo in the paper reviewed above.

I. Opatowski (Ann Arbor, Mich.).

Nadolschi, Victor L. Sur le mouvement d'un solide à masse variable dans un milieu résistant. Ann. Sci. Univ. Jassy. Sect. I. 27, 531–535 (1941).

The solution of $mdv/dt=a-bv^2$ is reduced to quadratures when m is a given function of t or a given function of v. In the reviewer's opinion, the author's physical interpretation is valid only if the variation of mass has velocity v just before joining the system. *P. Franklin.*

Casale, Ambrogio. Dimostrazione fatta coi vettori e generalizzazione di un teorema del Siacci. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) 9(78), 275–278 (1945).

For a skew trajectory, the force on a point is decomposed into three components, one along the tangent, one along the binormal and the third along a line passing through a fixed point. *From the author's summary.*

Casale, Ambrogio. L'unica forza centrale posizionale che fa descrivere una conica a un punto. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) 8(77), 437–440 (1944).

Masotti, Arnaldo. Sul moto di un punto vincolato ad una linea piana nel quale è costante la intensità della reazione. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) 6(75), 211–231 (1942).

Masotti, Arnaldo. Sulla dinamica di un punto vincolato ad una linea cicloidale. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) 6(75), 373–386 (1942).

Kraus, Wilhelm. Beweis einiger neuer Ungleichungen und Sätze in der äusseren Ballistik. I. Deutsche Math. 7, 39–50 (1942).

Some inequalities for data at the summit of a plane trajectory are derived, which reduce to equalities only in the vacuum case. In particular, it is shown that

$$\tfrac{1}{2}gt_*^2\leqq\tfrac{1}{2}x_*\tan\omega^0\leqq z_*\leqq\tfrac{1}{2}(x^0-x_*)\tan|\omega^0|,$$

$$\frac{x_*}{t_*}\geqq v_*\geqq\frac{x^0-x_*}{t^0-t_*},\quad x_*\leqq\tfrac{1}{2}x^0+z_*\{\cot\omega^0-\cot|\omega^0|\},$$

where the upper index 0 marks the point of fall and the lower index * marks the summit. A very general form is retained for the air resistance function. The paper is to be continued. *A. A. Bennett* (Providence, R. I.).

Lichnerowicz, André. Sur la transformation des équations de la dynamique. C. R. Acad. Sci. Paris 223, 649–651 (1946).

M. T. Y. Thomas a résolu le problème de la transformation des équations de la dynamique dans le cas de systèmes à liaisons indépendantes du temps [J. Math. Phys. Mass. Inst. Tech. 25, 191–208 (1946); ces Rev. 8, 102]. L'auteur annonce la résolution du même problème dans le cas de systèmes à liaisons dépendantes du temps mais dont la force vive et les forces généralisées ne dependent pas explicitement du temps. Les equations du mouvement d'un système D peuvent s'écrire

$$\frac{d^2x^\alpha}{dt^2}+\Gamma^{\alpha}_{\gamma\beta}\frac{dx^\gamma}{dt}\frac{dx^\beta}{dt}+a^{\alpha}_{\beta}\frac{dx^\beta}{dt}=\varphi^\alpha,$$

où $a^\alpha_\beta=2\partial_{[\alpha}a_{\beta]}$; la force vive de D a la forme

$$2T=g_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt}+2a_\alpha\frac{dx^\alpha}{dt}+A,$$

où les $g_{\alpha\beta}$, a_α et A ne dépendent que des x^α. Il y a des formes analogues pour le système E. Dans trois théorèmes l'auteur donne les conditions pour que toute trajectoire de E soit une trajectoire de D. Il résulte que dans le cas général (contrairement au cas restreint) les trajectoires de D et E sont identiques. *J. Haantjes* (Amsterdam).

Hydrodynamics, Aerodynamics, Acoustics

Høiland, Einar. The developed form of the dynamic boundary condition with applications. I. Two-dimensional motion of the homogeneous and incompressible ideal fluid. Arch. Math. Naturvid. 46, no. 2, 19–45 (1943). [MF 12978]

The author investigates the dynamic properties of the motion of an ideal two-dimensional incompressible fluid of homogeneous density in the neighborhood of a surface of discontinuity of the flow. The author states that it is too difficult to give a precise, or quantitative, analytical solution to the problem and investigates only some qualitative properties. At least formally an extension is given of Helmholtz' famous principle of the conservation of vortices. The results are applied to some examples: a finite line of discontinuity in the flow, a closed line of discontinuity and lines of discontinuity of infinite length. A line of finite discontinuity is instantaneously distorted, while a closed curve discontinuity becomes distorted after a finite time. Lines of discontinuity of infinite length are distorted with time into periodic spirals. *A. Gelbart* (Syracuse, N. Y.).

Poncin, Henri. Sur l'écoulement des fluides qui présentent une surface libre à pression constante. C. R. Acad. Sci. Paris 218, 102–104 (1944). [MF 13457]

By the method of analytic continuation the condition for the formation of free boundaries of constant pressure in the flow of an incompressible fluid is obtained. Properties of the free boundaries are also given. The presentation is very brief and no details are given. *A. Gelbart.*

Sbrana, Francesco. Sul moto di un solido immerso in un fluido. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) 6(75), 570–574 (1942).

Agostinelli, Cataldo. Applicazione del metodo delle immagini alla determinazione del moto liquido piano in una corona circolare in cui si formino dei vortici puntiformi. Problemi elettrostatici corrispondenti. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) **6(75)**, 669–689 (1942).

Reiner, M. The coefficient of viscous traction. Amer. J. Math. **68**, 672–680 (1946).

In a previous paper [same J. **67**, 350–362 (1945); these Rev. **7**, 44], the author derived a general formula for the relation between the stress and velocity strain tensors for an isotropic viscous fluid. Furthermore, he studied the case of the so-called Reynolds fluid. In the present paper, he considers a more general fluid, the Trouton fluid. The experimental work of Trouton [Proc. Roy. Soc. London. Ser. A. **77**, 426–440 (1906)] is briefly discussed and a relation between the stress and velocity strain tensors is obtained. Finally, the author raises the question of the legitimacy of the accepted relation between the stress and velocity strain tensors for an ordinary viscous fluid. In fact, he contends that the conventional Navier-Stokes equations are valid only if the divergence of the velocity vector vanishes. Since part of the author's argument is based upon experimental results, the reviewer feels unable to comment upon this contention. *N. Coburn* (Ann Arbor, Mich.).

Kampé de Fériet, Joseph. Sur la décroissance de l'énergie cinétique d'un fluide visqueux incompressible occupant un domaine plan borné. C. R. Acad. Sci. Paris **223**, 1096–1098 (1946).

The author establishes the inequality

$$E \leqq E_0 \exp(-8\nu K_C^{-1} t)$$

for the kinetic energy E of an incompressible viscous fluid in motion in a two-dimensional region S bounded by a regular closed curve C. Here ν is the coefficient of viscosity divided by the density and $K_C = 4\pi^{-2}\int_S\int_S(\log PQ)^2 dP dQ$. This is a refinement of a result of Leray [J. Math. Pures Appl. (9) **13**, 331–418 (1934)], obtained by an independent elementary argument. *J. W. Calkin* (Houston, Tex.).

Hamel, Georg. Eine komplexe Form der ebenen Bewegungsgleichungen zäher, inkompressibler Flüssigkeiten. Abhandlungen zur Hydrodynamik. X. Abh. Preuss. Akad. Wiss. Math.-Nat. Kl. **1941**, no. 2, 11 pp. (1941).

[For part IX cf. Z. Angew. Math. Mech. **21**, 129–139 (1941); these Rev. **3**, 92.] The momentum theorem for a viscous fluid is expressed by

$$\int \rho \frac{\partial \bar{v}}{\partial t} dV + \oint \rho \bar{v}(\bar{v}\bar{n}) dS = -\oint p\bar{n} dS + \oint \tau_n dS.$$

The Stokes-Navier equations are normally obtained by transforming the surface integrals into volume integrals. The author introduces the stream function ψ for two-dimensional incompressible motion and transforms the first integral into a surface integral. Simple transformation then leads to the expressions

$$\oint \{-\psi_t + \psi_x\psi_y + \rho^{-1}X_{(y)}\}dx + \{\psi_y^2 + p\rho^{-1} - \rho^{-1}X_{(x)}\}dy = 0,$$
$$\oint \{-\psi_x^2 - p\rho^{-1} + \rho^{-1}Y_{(y)}\}dx + \{-\psi_t - \psi_x\psi_y - \rho^{-1}X_{(y)}\}dy = 0.$$

These equations are satisfied by means of two functions φ and χ; thus $\varphi_x = -\psi_t + \psi_x\psi_y + \rho^{-1}X_{(y)}$, etc. The pressure term is easily eliminated and the equations for ψ, φ, χ become

$$\psi_y^2 - \psi_x^2 - 4\nu\psi_{xy} = \varphi_y + \chi_x,$$
$$2\psi_x\psi_y + 2\nu(\psi_{yy} - \psi_{xx}) = \varphi_x - \chi_y,$$
$$\varphi_x + \chi_y + 2\psi_t = 0.$$

The first two equations can be combined into one complex equation of the form $2\psi_z^2 + 4\nu i\psi_{zz} = -i(\varphi - i\chi)_z$. It is to be noted that the elimination of the pressure does not require the existence of third derivatives of the velocity components.

The equations can be further simplified for steady flow. With $\varphi = \omega_y$, $\chi = -\omega_x$ and $\omega - 2\nu i\psi = w$, the equations reduce to $w_{zz} + \frac{1}{16}\nu^{-2}\{(w - \bar{w})_z\}^2 = 0$. The extension to three dimensions is sketched and some applications discussed.

H. W. Liepmann (Pasadena, Calif.).

Cârstoiu, Ion. Généralisation des formules de Helmholtz et de Cauchy pour un fluide visqueux incompressible. C. R. Acad. Sci. Paris **223**, 1095–1096 (1946).

The author generalizes the equations of Helmholtz by showing that

$$\frac{d\xi}{dt} = \xi\frac{\partial u}{\partial x} + \eta\frac{\partial u}{\partial y} + \zeta\frac{\partial u}{\partial z} + \nu\Delta\xi, \cdots,$$

and shows that an integral of these equations gives a generalization of the equations of Cauchy. *C. C. Torrance.*

Weber, Hans R. Über die biharmonische Differentialgleichung als Strömungsgleichung zäher, inkompressibler Flüssigkeiten in der Ebene. Deutsche Math. **7**, 50–55 (1942).

This paper considers the biharmonic equation for the stream function which is obtained after eliminating the pressure and neglecting the inertial terms in the equations of plane motion of a viscous incompressible fluid. It is remarked that this equation, with the customary boundary conditions for a viscous fluid, is insufficient to determine the flow uniquely if the flow region is multiply connected. This is demonstrated in the example of the flow between two coaxial cylinders. The author then derives an integro-differential equation for the stream function, in which the pressure again does not appear, but which does determine the flow uniquely in any multiply connected region. In the proof of this fact a unique solution of the original equations of motion is assumed. *D. Gilbarg* (Bloomington, Ind.).

Robin, Louis. Complément à l'étude des mouvements d'un liquide visqueux illimité. J. Math. Pures Appl. (9) **23**, 91–96 (1944). [MF 15743]

The paper is an extension of a previous discussion by Leray and Robin [C. R. Acad. Sci. Paris **205**, 18–20 (1937)] and Leray [Acta Math. **63**, 193–248 (1934)].

H. W. Liepmann (Pasadena, Calif.).

de Colle, Licia. Teorema di minimo relativo a fluidi viscosi generali. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) **6(75)**, 343–352 (1942).

Toraldo di Francia, Giuliano. Sui moti di un liquido viscoso fra paretti cilindriche coassiali. Caso stazionario. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) **6(75)**, 575–586 (1942).

Klose, Alfred. Theorie der Luftkräfte bei verschwindender Reibung. Abh. Preuss. Akad. Wiss. Math.-Nat. Kl. **1941**, no. 9, 50 pp. (1941).

This is an attempt to construct a theory of viscous incompressible flow about solid bodies, with separation and dead-

water "wakes," which in the limiting case of vanishing viscosity will coincide with the theory of discontinuous potential flow (e.g., the Helmholtz-Kirchhoff theory). The concept of the boundary layer is adopted; in the limiting case this layer becomes a vortex sheet. The author admits the possibility, even in the limiting case, of finite wakes [see also Schmieden, Luftfahrtforschung **17**, 37–46 (1940); these Rev. **2**, 171, and Kolscher, Luftfahrtforschung **17**, 154–160 (1940); and especially these Rev. **4**, 59]. He obtains a formula for the force on the body, involving both viscous and pressure forces; in the limit the former disappear. [The remaining expression appears to be consistent with the Helmholtz-Kirchhoff theory, but conclusions are drawn for cases of finite wakes, which seems to involve a contradiction.] In a supplementary investigation it is shown that for finite Reynolds number R separation can occur only in a region of rising pressure. This result, the author states, has been assumed without proof in the Prandtl boundary-layer theory. As $R\rightarrow\infty$, the separation point moves to the point of minimum pressure, in agreement with a result previously deduced in the present paper. As a numerical example, he treats the case of flow about a sphere with separation at the equator. [The calculation is very crude and does not actually involve some of the critical aspects of the theory. The partial agreement of the results with observed sphere drag may be fortuitous.]

W. R. Sears (Ithaca, N. Y.).

Couchet, Gérard. Remarques géométriques sur la résultante des efforts agissant sur un profil en rotation uniforme. C. R. Acad. Sci. Paris **223**, 974–976 (1946).

Geometrical results are obtained, in two-dimensional airfoil theory for a uniformly rotating airfoil, between the location of the center of rotation and the resultant force on the airfoil. These results are of the same kind as the relations for the airfoil in uniform translation for which variations of the angle of attack produce the metacentric parabola as envelope of the resultant forces. The paper also corrects some misprints in two earlier notes by the author [same C. R. **221**, 280–282 (1945); **222**, 170–171 (1946); these Rev. **7**, 343; **8**, 109]. *E. Reissner* (Cambridge, Mass.).

Bergman, Stefan. Methods for determination and computation of flow patterns of a compressible fluid. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1018**, 71 pp. (8 plates) (1946).

The author's generalisation to compressible subsonic flow of the method of finding the stream function in incompressible flow as the imaginary part of a function of a single complex variable has been published in the same Tech. Notes, nos. **972**, **973** [these Rev. **7**, 342, 343]. As this method requires lengthy computations, the present paper discusses in detail the performance of these computations. The operations are divided into two groups, those which need only be carried out once for all and can then be tabulated, and those which have to be repeated in every individual case. A description is given concerning the performance of the necessary computations on machines which operate with punched cards.

There are four appendices which amplify or complete certain points in connection with Note **972**. The fourth appendix deals with the derivation of the complex potential in the hodograph plane for a symmetrical Joukowski profile.

L. M. Milne-Thomson (Greenwich).

Sauer, R. Method of characteristics for three-dimensional axially symmetrical supersonic flows. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **1133**, 24 pp. (11 plates) (1947).

Translation of Zentrale für Wissenschaftliches Berichtwesen über Luftfahrtforschung, Forschungsber. no. **1269** (1940).

Göthert, B. Plane and three-dimensional flow at high subsonic speeds. (Extension of the Prandtl rule). Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **1105**, 15 pp. (2 plates) (1946).

Translation of Lilienthal Gesellschaft für Luftfahrtforschung, Bericht no. **127**, pp. 97–101.

Feinsilber, A. M. Reduction of boundary layer equations for gases to the type of thermal conductivity equation. C. R. (Doklady) Acad. Sci. URSS (N.S.) **51**, 501–504 (1946).

The author describes a method which allows the approximate computation of boundary layer flow for a compressible fluid from a known incompressible solution. The ratio of the viscosity coefficients $z=\mu/\bar{\mu}$, where $\bar{\mu}$ denotes the value in the free stream, is chosen as dependent variable. The independent variables X and Y are defined by

$$X=m^{\frac{1}{2}}\int_0^x \bar{\mu}\bar{\rho}k^{-1}z_0^{-\frac{1}{2}n}dx, \quad Y=\psi(x, y).$$

Bars indicate the values in the free stream;

$$m=(\gamma+1)/(\gamma-1)a_0^2$$

is constant (γ, ratio of specific heats; a_0, velocity of sound at the stagnation point); n denotes the exponent of the viscosity-temperature relation (e.g., for air $n\doteq 0.76$); z_0 is the value of z at the wall; k is a constant to be determined later; ψ is the stream function. Using these variables the equations of motion become

$$\partial z/\partial X=k(z_0^\beta-z^\beta)^{\frac{1}{2}}z^{-\beta+1}\partial^2 z/\partial Y^2, \qquad \beta=1/n,$$

with the boundary condition $z=z_0=\{1+\frac{1}{2}(\gamma-1)M^2\}^n$, M the Mach number. This procedure is similar to the transformation given by von Mises. The author shows now that incompressible flow furnishes a similar equation with approximately the same coefficient if the velocity in the free stream is chosen as $\bar{u}=\{2[(1+\frac{1}{2}(\gamma-1)M^2)^n-1]\}^{\frac{1}{2}}$. Consequently solutions of the compressible case can be obtained from the incompressible case. The procedure is illustrated by a few examples. *H. W. Liepmann* (Pasadena, Calif.).

Sestini, Giorgio. Azioni dinamiche esercitate da una corrente traslocircolatoria su un arco di circonferenza con una sorgente eccentrica. Boll. Un. Mat. Ital. (2) **5**, 220–227 (1943).

Moresi, Maria Vittoria. Lo strato limite attorno ad un profilo ellittico. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) **6(75)**, 657–668 (1942).

Moresi, Maria Vittoria. Resistenza di un profilo ellittico investito da una corrente traslatoria a grandi numeri di Reynolds. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) **7(76)**, 103–114 (1943).

Cabella-Lattuada, Giulia. Resistenza incontrata da un filo sottilissimo traslante longitudinalmente in un liquido. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) **9(78)**, 21–34 (1945).

Konakov, N. K. A new formula for the coefficient of resistance in smoothtubes. C. R. (Doklady) Acad. Sci. URSS (N.S.) 51, 505–508 (1946).

The author develops an expression for the resistance in the case of axially symmetric turbulent flow. The resulting simple formula is compared with the well-known experiments of Nikuradze and good agreement is found for a large range of Reynolds numbers. *H. W. Liepmann.*

Temple, G., and Yarwood, Jennifer. Compressible flow in a convergent-divergent nozzle. Ministry of Aircraft Production [London], Aeronaut. Res. Council, Rep. and Memoranda no. **2077** (6347), 16 pp. (1942).

Exposition of results of Ringleb [Z. Angew. Math. Mech. **20**, 185–198 (1940); these Rev. **2**, 169].

Laitone, Edmund V. Exact and approximate solutions of two-dimensional oblique shock flow. J. Aeronaut. Sci. **14**, 25–41, 58 (1947).

Høiland, Einar. On the wave motions in sliding layers with internal static stability. Arch. Math. Naturvid. **47**, no. 3, 41–72 (1943). [MF 12974]

The author investigates the wave motion and the stability conditions of a hydrodynamic system consisting of two fluid layers separated by a surface of discontinuity in the velocity field, and, in general, in the mass field. The fundamental translational velocity has the same direction and magnitude, and is parallel to, the surface of discontinuity in each layer. The system is bounded by rigid planes which are also parallel to the surface of discontinuity.

The solutions representing permanent waves in such a system with the corresponding frequency equation determining the frequencies or the velocity of such waves as a function of the wavelength have been investigated [V. Bjerknes, J. Bjerknes, H. Solberg, T. Bergeron, Hydrodynamique Physique, Paris, 1934, chap. 9]. The consequences drawn from this frequency equation have been rather incomplete and in some instances wrong, while the question of the validity of the solutions for a complex value of the velocity of propagation seems not to have been raised. On the assumption that they were valid the stability criterion was deduced. The author points out that this assumption is not justified when internal stability occurs in the layers, establishes new criteria for stability and obtains results essentially different from the old ones. The fluid is assumed to be incompressible and homogeneous.
A. Gelbart (Syracuse, N. Y.).

Fjeldstad, J. E. Tidal waves of finite amplitude. Astrophys. Norvegica **3**, 223–245 (1941).

The problem of tidal waves is greatly simplified by the author under the assumption that the vertical acceleration can be neglected. Then the fluid velocity is always parallel to the undisturbed water surface and this velocity is the same at every point on a vertical line. The problem is then reduced to the similar problem of nonsteady two-dimensional motion of a gas. This reduction is well-known. The author considers plane wave motion in a canal; therefore the calculation is further reduced to that of propagation of plane waves of large amplitude in a gas. The latter problem was first treated by Riemann and has been extensively developed during recent years. The author reproduces such well-known features as the steepening of wave fronts by using both Eulerian and Lagrangian coordinates. However, there is doubt whether the basic assumption of negligible vertical acceleration can be expected to hold under such extreme situations as a steep wave front. *H. S. Tsien.*

Taylor, G. I. The air wave surrounding an expanding sphere. Proc. Roy. Soc. London. Ser. A. **186**, 273–292 (1946).

The flow of the air surrounding a sphere which expands with constant velocity beginning with the radius zero is found by assuming that the radial component u of the velocity and the sound speed c depend only on r/t, r being the distance from the center of the sphere, t the time. The equations of isentropic compressible fluid flow can then be reduced to an ordinary differential equation of first order whose solutions are easily discussed. It is, in particular, shown that, even for arbitrarily small velocities of the expanding sphere, the resulting air wave is indicated by a shock. Comparison of the results with those obtained by the linear theory of sound shows good agreement except for the pressure distribution near the surface of the sphere.
K. O. Friedrichs (New York, N. Y.).

Bordoni, Piero Giorgio. The conical sound source. Ricerca Sci. **15**, 250–251 (1945).

Italian abstract of a paper which appeared in J. Acoust. Soc. Amer. **17**, 123–126 (1945); these Rev. **7**, 218.

Barducci, I. Effetto della viscosità e della conduzione termica in un risuonatore acustico. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **1**, 764–774 (1946).

MATHEMATICAL PHYSICS

Optics, Electromagnetic Theory

Thomescheit, Alfred. Trigonometrische Durchrechnung von Strahlen bei dezentrierten optischen Systemen aus sphärischen Flächen. Z. Instrumentenkunde **61**, 201–208 (1941).

Decentration of a surface is defined as a rotation of the sphere about its intersection with the ideal optical axis of the system. When more than one surface is decentered, consideration is limited to the case in which the centers of curvature of the surfaces all lie in the same plane through the optic axis, and to rays lying in this plane. Familiar exact triangulation procedures are adapted to take the decentration into account. Formulae for meridional and sagittal foci are similarly modified. A numerical example is given. *A. J. Kavanagh* (Buffalo, N. Y.).

Durand, Émile. Marches paraxiales (dioptrique du 1[er] ordre). Rev. Optique **23**, 215–221 (1944).

This is the second in a series of three papers by the same author [cf. the same Rev. **23**, 91–104 (1944); these Rev. **7**, 269, and the following review]. Two paraxial rays are traced through the optical system: one from the axial object point and intersecting the entrance pupil at unit height, the other through the axial point of the entrance pupil plane and having unit paraxial inclination to the optic axis. Several quantities such as the Lagrange-Helmholtz invariant and the coefficients of the transformation from object space to

image space can be expressed simply in terms of the data of these rays. A list of errata in the first paper of the series is appended. *A. J. Kavanagh* (Buffalo, N. Y.).

Durand, Émile. Aberrations du troisième ordre dans les systèmes centrés et aberrations chromatiques. Rev. Optique 24, 137–150 (1945).

This is the last in a series of three papers by the same author [cf. the preceding review]. A third-order expansion is derived for optical path length along a ray through the system, starting from the object plane and ending at any desired plane in the image space. The expansion is in terms of paraxial data described in the previous paper. Application of Fermat's principle gives focal characteristics in the image space and leads to expressions for the third-order aberrations. A similar treatment, to first order, gives expressions for axial and lateral color. A numerical example is included. *A. J. Kavanagh* (Buffalo, N. Y.).

Duffieux, P. Michel, et Lansraux, Guy. Les facteurs de transmission et la lumière diffractée. Rev. Optique 24, 65–84, 151–160, 215–230 (1945).

The methods of Fourier transform theory are applied to discuss the imaging of coherently and incoherently lit objects by optical systems of small numerical aperture. If $F(x, y)$ represents the complex wave-displacement in the plane of the exit pupil of such a system, $G(u, v)$ the corresponding displacement in its image-plane, then with a suitable choice of units

$$G(u, v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(x, y)e^{-2\pi i(ux+vy)}dxdy,$$

$$F(x, y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(u, v)e^{2\pi i(ux+vy)}dudv;$$

the intensity in the diffraction image is proportional to $E(u, v) = |G(u, v)|^2$.

If $I(u, v)$ is the intensity distribution in an incoherently lit object, transferred by geometrical (Gaussian) imaging into the image-plane, the corresponding physical intensity distribution in the image-plane is given by

$$\Pi(u, v) + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} I(u', v')E(u-u', v-v')du'dv',$$

where $E(u, v)$ is the intensity distribution in the image by the system of a point-source at the Gaussian conjugate of (u, v). Then the double Fourier transform or space-frequency spectrum $T[\Pi(u, v)]$ of the physical image is connected with that of the object, namely $T[I(u, v)]$, by the relation $T[\Pi(u, v)] = T[I(u, v)]T[E(u, v)] = D(x, y)T[I(u, v)]$, where

$$D(x, y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E(u, v)e^{2\pi i(ux+vy)}dudv;$$

$D(x, y)$ is thus a "transmission factor" which characterises the imaging properties of the system.

In order to study the intensity $E(u, v)$ along the line $v=0$ in the image-plane, the authors write

$$f(x) = \int_{-\infty}^{\infty} F(x, y)dy, \quad \gamma(u) = G(u, 0) = \int_{-\infty}^{\infty} f(x)e^{-2\pi iux}dx$$

and call $f(x)$ the equivalent linear pupil of $F(x, y)$. They then set $E(u, 0) = |\gamma(u)|^2 = \eta(u)$ and define

$$d(x) = \int_{-\infty}^{\infty} D(x, y)dy = \int_{-\infty}^{\infty} \eta(u)e^{2\pi iux}du, \quad \delta(x) = D(x, 0).$$

If the object is an incoherently lit sinusoidal grating $I(u, v) = I(u) = A_0 + A\cos 2\pi ux$ of visibility A/A_0, then the visibility v' of the image is given by the equation $v' = (|\delta(x)|/\delta(0))v$. In the case of a coherently lit grating, the expression for the visibility becomes $v' = |F(x, 0)/F(0, 0)|^2v$.

The remainder of the paper is of less practical interest. In the second section, equations are derived connecting the derivatives of $f(x)$, the factor $d(x)$ and the general distribution of the diffracted light surrounding the image, but the arguments are purely formal and the majority of the results are not applicable to systems limited by circular stops in the usual manner. The third section considers in detail some examples of error-free exit pupils of different shapes and also an example of a circular exit pupil shaded off at the margin; it is shown that the effect of the shading is to redistribute the widely diffracted light so as to bring it nearer to the central part of the image. The fourth and fifth sections return to the two-dimensional functions $F(x, y)$, $D(x, y)$, $G(u, v)$. In the fourth section, general relations between them are given, analogous to those derived for $f(x)$ and $d(x)$ in the second section. The arguments are purely formal, and some of the difficulties ignored appear fundamental to the reviewer. The fifth section is devoted to an investigation of the properties of $D(x, y)$ in the practically important case of a finite exit-pupil and to the examination of particular cases. *E. H. Linfoot* (Bristol).

Bertein, François. Sur les imperfections de forme dans les instruments d'optique électronique. C. R. Acad. Sci. Paris 224, 106–107 (1947).

Vvedenskiĭ, B. A., and Ponomarev, M. I. Application of the methods of geometrical optics to the determination of the trajectories of ultra-short radio waves in a non-homogeneous atmosphere. Bull. Acad. Sci. URSS. Cl. Sci. Tech. [Izvestia Akad. Nauk SSSR] **1946**, 1201–1210 (1946). (Russian)

Polara, V. Sulla teoria dei reticoli di diffrazione. Atti Accad. Peloritana. Cl. Sci. Fis. Mat. Nat. (3) **4**(46), 116–131 (1944).

Françon, Maurice. Image d'un petit objet rectangulaire parfaitement transparent par la méthode du contraste de phase. Rev. Optique 25, 257–266 (1946).

Zernike [Physica 9, 974–986 (1942)] showed that it is possible to observe a small transparent object whose refractive index differs slightly from that of the surrounding medium by the method of contrast of phase. The essential idea is as follows. The bright image of the slit makes it impossible to observe the small object mounted in it. But the diffraction pattern of the slit has a strong central maximum, that of the small object a zero central minimum. Masking the centre of the diffraction pattern cuts down the brightness of the image of the slit without affecting the brightness of the image of the small object, which thereby becomes visible.

In the present paper, a mathematical theory of the method is developed for the case when the small object is a long thin rectangular lamina. The theory depends in the usual way on Fourier's integral. *E. T. Copson* (Dundee).

Bannon, J. A study of the reflection of light in the case of three homogeneous, isotropic, non-conducting media in successive contact. J. Proc. Roy. Soc. New South Wales 79, 101–115 (1946).

The author studies the reflection of light if a plane monochromatic wave is incident upon two plane parallel surfaces separating three homogeneous media. The media are assumed to be isotropic and nonconducting. The special case of normal incidence is well known theoretically and has practical significance for the construction of nonreflecting thin films. In this paper the author presents the solution for oblique incidence and thus obtains a complete theory of a thin film. By summing up the amplitudes of the total set of multiply reflected waves explicit formulae are derived for the parallel and normal components of the reflected electromagnetic vectors. These formulae are examined as to the influence of the thickness of the film and of the angle of incidence on the reflected wave. *R. Luneberg.*

Abelès, Florin. Nouvelles formules relatives à la lumière réfléchie et transmise par un empilement de lames à faces parallèles. C. R. Acad. Sci. Paris 223, 891–893 (1946).

Abelès, Florin. Formules de récurrence et deux théorèmes relatifs à la lumière réfléchie et transmise par un empilement de lames minces à faces parallèles. C. R. Acad. Sci. Paris 223, 1112–1114 (1946).

Locatelli, Piero. Sui modelli elastici di campi elettromagnetici. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) 5(74), 548–555 (1941).

Craggs, J. W. The determination of capacity for two-dimensional systems of cylindrical conductors. Quart. J. Math., Oxford Ser. 17, 131–137 (1946).

A method is presented for determining the capacity of systems of long parallel cylindrical conductors with electrostatic charges. The principal illustration used is that of a charged wire placed symmetrically between two parallel connected wires having the same diameter as the first. Unknown functions of the angular coordinate are introduced to represent the charges on the surfaces. These functions are replaced by trigonometric series representations with unknown coefficients. The electrostatic potentials at points on the surfaces are then expressed in terms of those coefficients and, using the fact that the potentials are constant over conducting surfaces, an infinite system of algebraic equations is obtained in those coefficients and the surface potential or the capacity. When those coefficients are eliminated the resulting equation takes the form of an infinite determinant set equal to zero, and from this equation approximate values of the capacity can be calculated.
R. V. Churchill (Ann Arbor, Mich.).

Craggs, J. W., and Tranter, C. J. The capacity of two-dimensional systems of conductors and dielectrics with circular boundaries. Quart. J. Math., Oxford Ser. 17, 138–144 (1946).

The method presented in the paper reviewed above is extended here to problems of determining the capacities of systems of cylindrical conductors which are imbedded in cylindrical dielectrics. A fictitious charge distribution on the surface separating two dielectrics is used to account for the difference in dielectric constants. The method is first applied to a system consisting of twin cables with each conductor surrounded by a coaxial dielectric sheath. This method of computing the capacity is an improvement over the authors' earlier method for this system [Quart. Appl. Math. 3, 380–383 (1946); these Rev. 7, 270]. The second example is that of a pair of parallel wires imbedded in a single cylinder of dielectric. *R. V. Churchill.*

Pinney, Edmund. The electrostatic field of two coplanar plates. Bull. Amer. Math. Soc. 52, 838–843 (1946).

In a recent paper N. Davy [Philos. Mag. (7) 36, 153–169 (1945); these Rev. 7, 179] discussed the problem of the electrostatic field about two thin infinitely long parallel coplanar metal plates of unequal width at equal and opposite potentials. The author obtains an expression for the field of such a system in the more general case in which the ratio of the charge per unit length on one plate to that on the other has a given value. The method used is a straightforward application of formulas developed in an earlier paper [Ann. of Math. (2) 47, 221–232 (1946); these Rev. 7, 451]. *M. C. Gray* (New York, N. Y.).

Davy, N. The field of a charged semi-infinite rectangular conductor parallel to an earthed infinite plane conductor or the flow from a thick-walled jet. Philos. Mag. (7) 37, 207–216 (1946).

Vvedenskiĭ, B. A., and Arenberg, A. G. The method of the Hertz vector in problems of practical electrodynamics. Bull. Acad. Sci. URSS. Cl. Sci. Tech. [Izvestia Akad. Nauk SSSR] 1946, 1211–1218 (1946). (Russian)

Gvosdover, S., and Lopukhin, V. Theory of single-circuit clystrons. Acad. Sci. USSR. J. Phys. 10, 275–284 (1946).

A plane parallel diode is assumed with a sinusoidal voltage applied between anode and cathode. From the differential equation of motion of a single representative electron, the integral equation for the total current density on the anode is deduced in terms of the applied voltage. For negligible space charge, this diode is now considered as the cavity with total electron stream I. Cathode and anode are then externally connected through inductance L and resistance R, so that a nonlinear oscillatory circuit is created with the equation

$$(1)\qquad (\omega^2/\omega_0^2)V_{xx}+\epsilon f(x, V, \epsilon)+V(x)=0,$$

where $x=\omega t$, ω is the free angular velocity, $\omega_0^2=1/LC$ and $\epsilon=R/\omega_0 L$. The voltage $V(x)$ is the diode voltage, for which an integral equation in terms of the electron stream exists. The solution of (1) is attacked in accordance with the method of successive approximations of M. Lindstedt and A. Liapounoff [see N. Kryloff and N. Bogoliuboff, Introduction to Non-Linear Mechanics, Ann. of Math. Studies, no. 11, Princeton University Press, 1943; these Rev. 4, 152]. Only the first correction term to the free frequency is derived in general form.

Application is made to the monotron, i.e., a single cavity klystron with impressed constant current density. The condition for self-excitation is established and both the free frequency and the amplitude of the voltage oscillation are computed. The maximum of power interaction between electron stream and cavity is also deduced. *E. Weber.*

Berg, T. G. Owe. Elementare Theorie des sphärischen Hohlraumresonators. Hochfrequenztech. Elektroak. 57, 56–60 (1941).

A theory for the spherical resonator is developed in a fairly straightforward fashion. The author limits his con-

siderations to the lowest mode of oscillation for $H_r=0$. The case of a spherical shell is discussed in a general way. In actually computing the frequency and attenuation for the lowest mode it is assumed that the shell is of infinite thickness. *R. S. Phillips* (New York, N. Y.).

Rydbeck, O. E. H. On the forced electromagnetic oscillations in spherical resonators. Ark. Mat. Astr. Fys. **32A**, no. 11, 18 pp. (1945).

The author presents a detailed study of the forced oscillations of a cavity oscillator in the form of a central sphere surrounded by a concentric spherical shell, with small loop oscillator between. The cases of transverse electric waves and of transverse magnetic waves are studied separately. In the former, for example, one has equations of the form

$$H_r=\left(k^2+\frac{\partial^2}{\partial r^2}\right)\Pi,\quad H_\theta=\frac{1}{r}\frac{\partial^2}{\partial r\partial\theta}(\Pi),\quad H_\varphi=0,$$

$$E_r=E_\theta=0,\quad E_\varphi=jzk\cdot\frac{1}{r}\frac{\partial}{\partial\theta}(\Pi),$$

with z as the intrinsic impedance, and $k=2\pi/\lambda$ as the complex wave number.

The paper is distinguished by its considerations of the coupling due to the exciting source. Explicit solutions of the wave equation are given, from which the author analyzes the relative effects of the central sphere and the shell on the wave transmission. *A. L. Foster.*

Kahan, Théodore. Recherches sur la propagation des ondes électro-magnétiques dans des espaces diélectriques doublement connexes (annulaires). I. Oscillations stationnaires dans une cavité toroïdale. Cahiers de Physique no. 1, 51–59 (1941).

Solutions of the wave equation are given in cylindrical coordinates for the interior of a finite length of an annular concentric cylindrical space. Assuming infinite electrical conductivity, the two types of modes, namely, transverse magnetic and transverse electric (in reference to the axis of the cylinders), are established involving the Bessel functions of the first and second kind. The resonant wave length is also computed. Extension to the interior of a toroid of circular cross section is indicated very briefly (though not in a plausible manner). *E. Weber* (Brooklyn, N. Y.).

Jouguet, Marc. Les oscillations électromagnétiques naturelles des cavités. Propriétés générales. Cavités sphériques. Rev. Gén. Électricité **51**, 318–323 (1942).

Jouguet, Marc. Les oscillations électromagnétiques naturelles des cavités ellipsoïdales. Rev. Gén. Électricité **51**, 484–487 (1942).

Buchholz, Herbert. Die Abstrahlung einer Hohlleiterwelle aus einem kreisförmigen Hohlrohr mit angesetztem ebenen Schirm. Arch. Elektrotechnik **37**, 22–32 (1943). [MF 15633]

The general problem considered by the author in this and the two succeeding papers is that of determining the field radiated into free space from the open end of a wave guide of finite length, when the guide is embedded in a semi-infinite perfect conductor and the open end of the guide lies in the interface between the conductor and free space.

In the present paper the application of Huyghens' principle to determine the radiated field from the open end of a guide in free space is first described, and then the principle is extended to the present problem. It is assumed throughout that the primary wave in the guide is transverse magnetic ($TM_{m,n}$ mode), and the electric and magnetic field vectors in free space for any pair of values m, n are derived. The author shows that the approximate nature of the Huyghens solution is due to the fact that the requisite boundary conditions at the interface and over the open end of the guide are satisfied by the electric but not by the magnetic field vectors. *M. C. Gray* (New York, N. Y.).

Buchholz, Herbert. Die Abstrahlung einer Hohlleiterwelle aus einem kreisformigen Hohlrohr mit angesetztem ebenen Schirm. II. Die Lösung des Abstrahlungsproblems für die TM-Welle auf der Grundlage der Maxwellschen Gleichungen. Arch. Elektrotechnik **37**, 87–104 (1943). [MF 15632]

[Cf. the preceding review.] In this part of the paper the general problem is attacked by an entirely different method, taking into account the reflected and diffracted waves generated at the open end of the guide. The principal problem is now the determination of the amplitudes of these secondary waves from the boundary conditions; they are obtained as solutions of a linear system of infinitely many equations in an infinite number of unknowns. Formally the solution is complete, but in practice the coefficients in the system of equations are too complicated to permit of more than very approximate methods of solution. An approximate solution is outlined on the simplifying assumption that the length of the guide is an integral number of half-wavelengths of the primary wave. *M. C. Gray.*

Buchholz, Herbert. Die Ausstrahlung einer Hohlleiterwelle aus einem kreisförmigen Hohlrohr mit angesetztem ebenen Schirm. III. Arch. Elektrotechnik **37**, 145–170 (1943). [MF 15631]

[Cf. the two preceding reviews.] The author first describes the basic characteristics of the radiated field as obtained in part II. Polar radiation patterns for primary waves of type $TM_{0,1}$, $TM_{1,1}$, and $TM_{2,0}$ are reproduced. A mathematical appendix then gives the derivation of series expansions by means of which the integrals occurring in the solution were calculated. *M. C. Gray.*

Pistolkors, A. A. Radiation from longitudinal slits in a circular cylinder. C. R. (Doklady) Acad. Sci. URSS (N.S.) **52**, 127–130 (1946).

The author obtains a formula for the radiated field from a system of identical longitudinal slits located arbitrarily (but with no relative axial displacement) on the surface of an infinitely long perfectly conducting circular cylinder, assuming that the field distribution over all the slits is the same. In the case of a single narrow slit curves are drawn showing the radiation pattern in the plane perpendicular to the axis of the cylinder for various values of the cylindrical radius. *M. C. Gray* (New York, N. Y.).

Borgnis, F. Die Fortpflanzungsgeschwindigkeit der Energie monochromatischer elektromagnetischer Wellen in dielektrischen Medien. Z. Physik. **117**, 642–650 (1941).

For a monochromatic wave exp $\{j(wt-hx_i)\}$, phase velocity is defined as $v=w/h$, while according to Rayleigh for a group of waves $(w\pm\epsilon)$ the group velocity is defined as $u=dw/dh(w)$ and can easily be shown to be the velocity of the resultant amplitude. The author demonstrates that this group velocity becomes identical with the velocity of propagation of the average field energy for guided waves. He

assumes field problems in which two Hertz vectors can be chosen in the direction of propagation, one for the electric and the other for the magnetic mode. The lateral (guide) boundaries define the eigenvalues of the propagation constants. He evaluates in general form the Poynting energy flow in the direction of propagation and the average field energy of the waves. The ratio gives the energy velocity, identical with the group velocity above. For the principal mode, with no field component in the direction of propagation, group and phase velocity become identical. He explains their difference in the other cases by the fact that not all the field energy is propagated, making $u<v$.

E. Weber (Brooklyn, N. Y.).

Gogoladze, V. G. On the propagation of electromagnetic waves in different media adjoining each other along a plane. Bull. Acad. Sci. URSS. Sér. Géograph. Géophys. [Izvestia Akad. Nauk SSSR] **10**, 115–120 (1946). (Russian. English summary)

In connection with the development of the theory of the propagation of radio-waves, Zenneck, Sommerfeld and other authors studied the problem of the rise of electromagnetic surface waves at a contact along a plane. The question of the existence of these waves leads to the research of the roots of some algebraic functions satisfying the conditions of the problem. The author shows in the present article that this function has no roots in the first leaf of Riemann's surface and in consequence there do not exist any electromagnetic surface waves in the present case.

Author's summary.

Fok, V. A. The field of a plane wave near the surface of a conductor. Bull. Acad. Sci. URSS. Sér. Phys. [Izvestia Akad. Nauk SSSR] **10**, 171–186 (1946). (Russian)

Ponomarev, M. I. Application of the "phase integral" method to the solution of the problem of propagation of radio waves around the globe. Bull. Acad. Sci. URSS. Sér. Phys. [Izvestia Akad. Nauk SSSR] **10**, 189–195 (1946). (Russian)

Ponomarev, M. I. The effect of refraction on the propagation of radio waves around the earth. Bull. Acad. Sci. URSS. Cl. Sci. Tech. [Izvestia Akad. Nauk SSSR] **1946**, 1219–1233 (1946). (Russian)

Feinberg, E. On the propagation of radio waves along an imperfect surface. Acad. Sci. USSR. J. Phys. **10**, 410–418 (1946).

Continuation of the author's papers in the same J. **8**, 317–330 (1944); **9**, 1–6 (1945); cf. these Rev. **7**, 99.

Colombo, Serge. Sur les conditions aux limites dans l'intégration de l'équation des télégraphistes. C. R. Acad. Sci. Paris **222**, 283–284 (1946). [MF 16002]

Let a voltage $F(t)$ be applied to one end of a transmission line and let the other end be grounded through an impedance. Operational calculus is used to write a formula for the transient voltage at the latter end in terms of the given impedance, the impressed voltage and the characteristics of the line.

R. V. Churchill (Ann Arbor, Mich.).

★**Josephs, H. J. Heaviside's Electric Circuit Theory.** Chemical Publishing Co. of N. Y. Inc., New York, 1946. viii+115 pp. $2.25.

A standard treatment of the subject. The claim in the preface and chapter 5 that the book is based "on a far-reaching theorem which, so far, appears to have escaped the notice of engineers," a theorem "reconstructed from the scattered papers of Heaviside" and which leads "not only to electric circuit theorems, but also to theorems involving Fourier and Bessel functions, elliptic functions, etc.," is remarkable in view of the content of the theorem. It states [p. 53] in somewhat disguised notation that

$$\int_{-\infty}^{\infty} f(x)d\sigma(x)=f(0)$$

if $\sigma(x)$ is the unit step function.

H. Pollard.

Zimmermann, Fritz. Drehstromunsymmetrieprobleme in Matrizendarstellung. Arch. Elektrotechnik **38**, 131–140 (1944).

The author presents a matrix formulation of the current distribution in a simply forked, not necessarily symmetrically loaded network. Suitable specializations yield familiar results in the case of various types (multiple pole) of short circuits.

A. L. Foster (Berkeley, Calif.).

Parodi, Maurice. Réseaux électriques et théorie des transformations. J. Phys. Radium (8) **7**, 94–96 (1946).

Given a system of n independent electrical circuits, let l_{ij}, r_{ij}, s_{ij} denote the total self inductance, resistance and reciprocal capacity common to i and j $(=1, \cdots, n)$, let $\dot{x}_i$ be the current in the ith circuit and let

$$T=\tfrac{1}{2}\sum_{i,k=1}^{n}\sum l_{ik}\dot{x}_i\dot{x}_k, \quad U=\tfrac{1}{2}\sum\sum s_{ik}x_ix_k, \quad F=\sum\sum r_{ik}\dot{x}_i\dot{x}_k$$

be the associated quadratic forms. The proper frequencies of the system are determined by the equation

$$|Tp^2+Fp+U|=0 \tag{1}$$

and the impedance at the boundary of the circuit i is written

$$Z(p)=\frac{|Tp^2+Fp+U|}{|Tp^2+Fp+U|_i}, \tag{2}$$

where the subscript i denotes the ith minor. The author determines the class of ensembles of n circuits "equivalent to," that is, having the same proper frequencies as (1), and similarly the class having the same impedance (2). In the former, for example, any star transform $T=A^*TA$, $U=A^*UA$, $F=A^*FA$ of T, U, F, where A is an arbitrary $n\times n$ matrix with transpose A^*, determines a system equivalent to (1).

A. L. Foster (Berkeley, Calif.).

Parodi, Maurice. Conditions suffisantes pour qu'une matrice puisse caractériser un réseau de self-inductances dont les mailles indépendantes sont couplées par self-inductances et inductances mutuelles. J. Phys. Radium (8) **7**, 269–270 (1946).

Clavier, A.-G. Application de la transformation de Laplace à l'étude des circuits électriques. Rev. Gén. Électricité **51**, 447–455 (1942).

Quantum Mechanics

de Broglie, Louis. Sur l'application du théorème des probabilités composées en mécanique ondulatoire. C. R. Acad. Sci. Paris **223**, 874–877 (1946).

It is noted that, if U and V are observables in quantum mechanics which cannot be measured simultaneously, more

exactly if the corresponding operators are not commutative, the concepts of bivariate probability distribution and conditional distribution must be treated with great care; the knowledge of U, that is, a measurement of U, has a profound effect on the distribution of U and V, and no bivariate density function can be expected to yield the conditional distributions in the usual way. It is shown how these concepts may be treated. *J. L. Doob* (Urbana, Ill.).

Groenewold, H. J. On the principles of elementary quantum mechanics. Physica 12, 405–460 (1946).

Yvon, Jacques. Sur une propriété commune aux opérateurs différentiels et intégraux. C. R. Acad. Sci. Paris 223, 311–312 (1946).

The author gives the following condition that a differential operator G be self-adjoint. Let

$$g(x, p) = \exp(-ipx/h)G\exp(ipx/h),$$

and let $F(x, p, k)$ be that solution of the partial differential equation $\partial F/\partial k = i\partial^2 F/\partial x\partial p$ for which $F(x, p, 0) = g(x, p)$. Then the condition is that $g^* = F(x, p, h)$. He then shows that a condition that an integral operator $\int N(x, y)f(y)dy$ be self-adjoint can be stated in nearly the same form. *O. Frink* (State College, Pa.).

Yvon, Jacques. Sur les rapports entre la théorie des mélanges et la statistique classique. C. R. Acad. Sci. Paris 223, 347–349 (1946).

Modifying the results of the note reviewed above, the author associates with each self-adjoint differential operator G a real function $\gamma(x, p)$ and with each self-adjoint integral operator $L(f) = \int N(x, y)f(y)dy$ a probability density $D(x, p)$, so that the mean $\bar{G}$ of G relative to L is definable as $\bar{G} = \int\int \gamma(x, p)D(x, p)dxdp$. This probability density is real-valued and uniquely defined, whereas other definitions of probability density in quantum mechanics previously proposed are not unique and are complex-valued. *O. Frink* (State College, Pa.).

Destouches, Jean-Louis, et Viard, Jeannine. Définition du minimum d'une fonction opératorielle. Minimum de l'opérateur force vive relative à un repère en translation. C. R. Acad. Sci. Paris 223, 610–612 (1946).

This note extends to wave mechanics the theorem of König, which depends on the result that the kinetic energy of a system of particles with respect to a frame of reference T, which is moving in translation relative to a fixed frame, is a minimum when T has its origin at the center of gravity of the system. To make the extension it is necessary to define what is meant by the minimum value of an operator-valued function of operators, and in particular of a quadratic function of operators. *O. Frink.*

Potier, Robert. Sur certaines identités où interviennent les produits de fonctions d'ondes. C. R. Acad. Sci. Paris 223, 651–653 (1946).

The author defines bilinear forms of two two-component spinors $\psi(x_1, y_1; z_1, t)$ and $\varphi(x_2, y_2; z_2, t)$ with multiple indices; ψ and φ transform in accordance with some representation of the Lorentz group. He then lists identities which relate the derivatives of the bilinear forms with those of the derivatives of the spinors. *A. H. Taub* (Seattle, Wash.).

Arnous, Edmond, et Colombo, Serge. Méthodes d'approximations. Sur l'équivalence de la méthode d'itération et de la méthode des moments. C. R. Acad. Sci. Paris 223, 850–852 (1946).

La méthode d'itération a été utilisée avec succès pour l'étude des noyaux légers. Elle est équivalente à la méthode des moments, obtenue par l'un de nous comme approximation de la méthode (rigoureuse) de la fonction caractéristique. Son originalité, à notre avis, est de permettre le calcul des niveaux d'énergie à partir d'une fonction d'onde qui, en principe du moins, peut être à peu près arbitraire. Elle s'oppose ainsi résolument aux autres méthodes d'approximation, dont l'objet essentiel est de rechercher des fonctions d'onde aussi voisines que possible des fonctions propres de l'hamiltonien H. Nous nous contenterons de souligner l'équivalence de la méthode d'itération et de la méthode des moments et de préciser les meilleures conditions de leur utilisation. *Extract from the paper.*

Coulson, C. A., and Rushbrooke, G. S. On the motion of a Gaussian wave-packet in a parabolic potential field. Proc. Cambridge Philos. Soc. 42, 286–291 (1946).

The authors investigate the quantum mechanical behavior of a harmonic oscillator when the initial position and momentum are given by Gaussian probability distributions. They find that the center of such a distribution or wave packet moves according to classical mechanics, and the width of the packet also varies classically if time is measured from an instant when the packet is "best-possible" according to the Heisenberg uncertainty relation, which happens four times during each oscillation. The special case of a free particle is also considered. *O. Frink.*

Caldirola, Piero. Forze non conservative nella meccanica quantistica. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) 2, 896–903 (1941).

It is shown that certain nonconservative systems containing forces which are linear in the velocities can formally be brought into the Lagrangian form if one substitutes for the time variable a suitable function of time. The author subjects this Lagrangian to the usual procedure of quantization. *F. W. London* (Durham, N. C.).

Corson, E. M. Second quantization and representation theory. Physical Rev. (2) 70, 728–748 (1946).

Using the framework of the Dirac-Jordan representation theory, the author develops the formalism of second quantization. First the transformation of the representations between two complete commuting sets of observables is carried through for a system of n indistinguishable particles and then the two cases of symmetric and antisymmetric wave functions are considered in detail. In the former case the dynamical variables are shown to be analogous to the action and angle variables of the harmonic oscillator and in the latter case they are shown to be related to the Pauli spin operators. In each case explicit expressions for the representations of operators which depend on the coordinates of only one particle and of two particles are given. The equivalence of this formalism and the methods of the ordinary coordinate-spin space is shown and the equations of connection are fully developed. Finally the theory is illustrated by applying it to derive the Fock-Dirac density matrix and the self-consistent field. *S. Kusaka.*

Brekhovskich, L. M. Radiation of gravitational waves by electromagnetic waves. C. R. (Doklady) Acad. Sci. URSS (N.S.) 49, 482–485 (1945).

The author obtains approximate solutions of the joint system of gravitational and electromagnetic field equations and uses these solutions to calculate an approximate expression for the loss of energy by an electromagnetic wave due to gravitational radiation. In a cylindrical type world it is found that there is no analogue for the Cherenkov effect and that the magnitude of the gravitational radiation is not large enough to be detected by experiment.

M. Wyman (Edmonton, Alta.).

Soh, Hsin P., and Wang, Mu H. Lorentz transformation of the field strength of an accelerating moving charge. Acad. Sinica Science Record 1, 431–437 (1945).

It is shown that the electric and magnetic field intensity of an accelerated charge can be obtained by successive applications of the Lorentz transformation.

C. Kikuchi (East Lansing, Mich.).

Humblet, Jean. Le champ électromagnétique multipolaire. C. R. Acad. Sci. Paris 223, 419–421 (1946).

The author gives formulas for the principal parts of the electric and magnetic fields due to a system of charged particles in motion, on the basis of his theory of electric and magnetic multipole moments [Physica 11, 91–99, 100–113 (1944); these Rev. 7, 180.] *O. Frink.*

Wang, K. C., and Cheng, K. C. A five-dimensional field theory. Physical Rev. (2) 70, 516–522 (1946).

With the assumption of a five-dimensional space-time the field equations and the interactions of the field and particles are obtained. In this theory the electrodynamics is in agreement with the classical theory.

From the authors' summary.

Novobátzky, K. F. Mehrkörperproblem in der Quantentheorie. Mat. Fiz. Lapok 48, 312–333 (1941). (Hungarian. German summary)

Im Rahmen einer allgemeinen Besprechung des quantentheoretischen Mehrkörperproblems werden die Schwierigkeiten, mit denen die Quantelung des elektromagnetischen Feldes behaftet ist, durch Einführung eines neuartigen Operatorenverfahrens behoben. Im Sinne einer korrekten Deutung der Diracschen Gleichung werden aus dem Ausdruck der Energiefunktion sowohl das elektrostatische, als auch das elektrodynamische Selbstpotential der geladenen Teilchen ausgemerzt, woraus sich ein klassich singularitätenfreier Hamiltonoperator für die Elektrodynamik ergibt.

Author's summary.

Racah, Giulio. On the self-energy of the electron. Physical Rev. (2) 70, 406–409 (1946).

The author makes the conjecture that the series expression obtained by Weisskopf [Physical Rev. (2) 56, 72–85 (1939)] for the self-energy of an electron which is derived by applying the "hole" theory and is of the form (1) $w=\sum_n w^{(n)}$, (2) $w^{(n)}\sim mc^2(e^2/(hc))^n\{\log(h/(mca))\}^n$, converges. He supports these conjectures by calculating the second approximation of the electrostatic self-energy in the hole theory and finds it to be in accord with (2) and in effect negative.

A. H. Taub (Seattle, Wash.).

Schönberg, Mario. Negative energy states of the electron. Anais Acad. Brasil. Ci. 18, 93–101 (1946). (Portuguese)

Eliezer, C. Jayaratnam. The hydrogen atom in a generalized classical electrodynamics. Physical Rev. (2) 71, 49–53 (1947).

Since the previous work of the author has shown that the classical Lorentz-Dirac equation for a radiating electron does not always have physically sensible solutions, he has generalized the theory by abandoning the usual assumption that the field of a moving electron is given by its retarded field. Instead a combination of retarded and advanced fields is taken and the field due to an electron F_{el} is written $F_{el}=F_{ret}+k(F_{ret}-F_{adv})$, where F_{ret} and F_{adv} are the retarded and advanced fields. The case $k=0$ coincides with the Lorentz-Dirac theory, the case $k=-\frac{1}{2}$ with the theory with no radiation damping; the case $k=-1$ has the equation of motion exactly the same as for the Lorentz-Dirac theory except that the sign of the radiation damping term is reversed. In this paper the solution of the generalized equation of motion for $k<-\frac{1}{2}$ is investigated for an electron moving in the field of a proton. Explicit solutions are not obtained but the asymptotic behaviour of the solutions is studied. For rectilinear motion, it is shown that either (i) the electron moves toward the proton with increasing velocity and collides with it, or (ii) the electron recedes from the proton with steadily diminishing velocity. In the two dimensional motion the equations permit the electron to spiral around the nucleus and fall into it.

S. Kusaka (Princeton, N. J.).

Harish-Chandra. The correspondence between the particle and the wave aspects of the meson and the photon. Proc. Roy. Soc. London. Ser. A. 186, 502–525 (1946).

In Kemmer's "particle" formulation of the meson equation, the wave function is a one-column matrix which satisfies a differential equation of the same form as Dirac's except that the corresponding matrix coefficients β_k obey different commutation relations. The present paper extends Kemmer's work by deriving general relations between the matrices in this formulation and the tensor wave functions in the usual "wave" representation of the meson theory, and in this way shows the complete equivalence of the two theories. This result is attained by introducing one-column vector matrices Γ_k^* whose products with the wave function $\Gamma_k^*\psi$ transform like the components of a relativistic vector. With the aid of these matrices numerous algebraic relations are derived for the β matrices; in particular, formulae for the spur of the product of any number of β_k's are derived both for the 10-row and 5-row representations corresponding to spin 1 and 0. In addition, they permit the introduction of nuclear interactions in an elegant way. Kemmer's formulation can be applied only to a particle with nonvanishing rest mass but it is shown here that a slight modification of the same formalism can be adapted to the case of zero rest mass. Finally, an especially convenient representation for the matrices is given in terms of the fundamental spinor matrices of the Lorentz transformation. *S. Kusaka.*

Fremberg, N. E. Some applications of the Riesz potential to the theory of the electromagnetic field and the meson field. Proc. Roy. Soc. London. Ser. A. 188, 18–31 (1946).

This paper gives a number of applications of a method due to M. Riesz [see, for example, B. B. Baker and E. T. Copson, The Mathematical Theory of Huygens' Principle, Oxford University Press, 1939, pp. 54–67; these Rev. 1, 315] of solving normal hyperbolic differential equations, in particular the wave equation. It is shown that the method

gives a simple derivation of many results of classical electrodynamics. Dirac's equation of motion for a radiating electron [same Proc. Ser. A. 167, 148–169 (1938)] is obtained. The method is further applied to the Proca-Yukawa wave equation $(\Box+\chi^2)u=f$ to obtain Bhabha's results [same Proc. Ser. A. 172, 384–409 (1939)] on the neutral meson field. *C. Kikuchi* (East Lansing, Mich.).

Markov, M. On a certain criterion of relativistic invariance. Acad. Sci. USSR. J. Phys. 10, 333–340 (1946).

If two points in space time are such that the vector joining them is space-like then the measurement of one physical quantity M at one of these points cannot react on the measurement of a quantity N at the second of these points. That is, the operators representing these quantities at the respective points must commute. The author applies this principle of "causal independence" to the operators representing the vector potential of an electromagnetic field and the Dirac-Hamiltonian for a system of n electrons interacting with this field. He uses the Dirac-Fock-Podolsky many time formalism to obtain a condition on the factors introduced by any "cut-off" process for the Fourier expansion of the vector potential. Introducing such cut-off factors is equivalent to modifying the commutation equations for the Fourier components of the vector potential. When these are written as $[A^{+}_{\nu\mu}(k_\mu), A^{-}_{\mu\gamma}(k_\gamma)]=c(k_\mu, \lambda)\delta_{\mu\gamma}$, instead of $[A^{+}_{\nu\mu}(k_\mu), A^{-}_{\mu\gamma}(k_\gamma)]=\delta_{\mu\gamma}$, the function $c(k, \lambda)$ cannot be arbitrary but must satisfy the condition referred to above.

The author shows that, if $c=e^{k^\mu\lambda_\mu}$, $\mu=1, 2, 3, 4$, this condition is violated and hence the principle of causal independence at two relatively space-like points does not hold. He interprets the result as follows. The introduction of the function $e^{k^\mu\lambda_\mu}$ in the commutation relations is equivalent to postulating a certain charge distribution for the electron, as may be seen from an application of the many time formalism. Signals are propagated across this charge distribution at a velocity greater than that of light and as a result the causal independence principle is violated. He then states that any choice of an integrable function for $c(k, \lambda)$ would determine a charge distribution and the same difficulty would then arise.

The author then reviews the singular function $c(k, \lambda)$ introduced in the Wentzel-Dirac limiting process and a generalization of this function. He also reviews the Heitler-Peng "calculation scheme" [Proc. Cambridge Philos. Soc. 38, 296–312 (1942)] in light of the causal independence principle and finds that this principle is satisfied.
A. H. Taub (Seattle, Wash.).

Petiau, Gérard. Sur des intégrales premières de la théorie de l'électron de Dirac. C. R. Acad. Sci. Paris 223, 269–270 (1946).

A. Proca has shown that, if ξ is a combination of the Dirac matrices α_1, α_2, α_3, α_4, then a certain operator $u(\xi)$ formed from ξ is a first integral of the Dirac equation for a free electron in the sense that

$$\frac{d}{dt}\int \psi^* u(\xi)\psi d\tau=0, \tag{1}$$

where ψ is a solution of the Dirac equation. The author seeks to extend this result in certain cases to an electron which is in an external electromagnetic field. The corresponding operator $U(\xi)$ now depends on the external field, and equation (1) becomes more complicated, its right member no longer being zero. *O. Frink* (State College, Pa.).

Petiau, Gérard. Sur les équations d'ondes des corpuscules de spin quelconque. II. J. Phys. Radium (8) 7, 181–184 (1946).

The general wave equation for a particle with higher spin derived by the author in an earlier paper [same vol., 124–128 (1946); these Rev. 8, 124] is considered for the special cases of spins 3/2 and 2. The wave equations are separated into parts to which correspond different values of the rest mass, and the wave functions are transformed from the spinor representation to a mixture of spinor and tensor representation for spin 3/2 and to a tensor representation for spin 2. *S. Kusaka* (Princeton, N. J.).

Petiau, Gérard. Sur les principes généraux d'une nouvelle théorie unitaire des champs. J. Phys. Radium (8) 7, 226–227 (1946).

A suggestion is made that a possible way of generalizing Einstein's gravitational equation is to replace the classical energy momentum tensor by a combination of tensor densities of the type obtainable from the Dirac theory, $\psi^+\gamma_A\psi$, $\psi^+_\mu\gamma_A\psi$, $\psi^+\gamma_A\psi_\nu$, $\cdots$, $\psi^+_{\mu_1\cdots\mu_p}\gamma_A\psi_{\nu_1\cdots\nu_q}$, where the Greek subscripts indicate differentiations with respect to the coordinates, and γ_A stands for any of the 16 linearly independent products of the Dirac matrices. The author hopes that in this way a unified theory of gravitational and electromagnetic phenomena can be obtained which takes into account the quantum nature of elementary matter and electricity.
S. Kusaka (Princeton, N. J.).

Proca, Alexandre. Sur les équations relativistes des particules élémentaires. C. R. Acad. Sci. Paris 223, 270–272 (1946).

Bloch, Léon. Sur certaines relations quadratiques de la théorie du photon. C. R. Acad. Sci. Paris 223, 1099–1100 (1946).

Shanmugadhasan, S. On the theory of spinning particles. Proc. Cambridge Philos. Soc. 43, 106–117 (1947).

Translational and rotational equations of motion are formulated in classical and quantum theory for a particle, and for a number of particles, with charge, dipole moment $Z_{\mu\nu}$, and spin angular momentum $\theta_{\mu\nu}$, interacting with an electromagnetic field or a generalised (meson) field. Following Dirac [Communications Dublin Inst. Advanced Studies. Ser. A, no. 1 (1943); these Rev. 7, 100] and Bhabha and Corben [Proc. Roy. Soc. London. Ser. A. 178, 273–314 (1941); these Rev. 3, 158], the author evaluates the flow of energy and momentum from a portion of the world tube of a particle; the work is symmetrical with respect to retarded and advanced fields and with respect to incoming and outgoing fields. The relation $Z_{\mu\nu}=C\theta_{\mu\nu}$, C constant, is a possible consequence of the postulate that $\theta^2=\theta_{\mu\nu}\theta^{\mu\nu}$ is constant even in regions of nonzero effective field $f_{\mu\nu}$. The effective field at the space-time point $\mathbf{z}$ is derived from a potential $\mathbf{A}(\mathbf{z})=\frac{1}{2}[\mathbf{A}_W(\mathbf{z}+\lambda)+\mathbf{A}_W(\mathbf{z}-\lambda)]$, where $\mathbf{A}_W$ is the Wentzel field and the λ-limiting process [Dirac, loc. cit.] is used. For a particle of mass M, rotational kinetic energy $\theta^2/4I$, momentum $\mathbf{p}$, and space-time coordinate $\mathbf{z}$, the Hamiltonian equation is

$$F\equiv-\tfrac{1}{2}\{(\mathbf{p}-e\mathbf{A}(\mathbf{z}))^2-(M+(\theta^2/4I)-\tfrac{1}{2}C\theta^{\alpha\beta}f_{\alpha\beta})^2\}$$
$$+\{M+(\theta^2/4I)-\tfrac{1}{2}C\theta^{\alpha\beta}f_{\alpha\beta}\}=0.$$

Similar methods and results occur in the extension of the theory to generalised fields whose potentials U_ν satisfy $(\Box+\chi^2)U_\nu=0$, where χ is a constant. Again the Wentzel field and the λ-limiting process are used. *C. Strachan.*

Villars, Felix. Ein Beitrag zum Deuteronproblem. Helvetica Phys. Acta 19, 323–354 (1946).

The lowest states of a two-nucleon system are investigated on the basis of the strong coupling meson theory, which implies the existence of isobar states of the nucleons. It is essential that the tensor forces are taken into account. The (experimentally known) energy difference between the ^{3}S- and ^{1}S-states of the deuteron can be obtained only if the excitation energy of the isobars is assumed greater than 200 Mev, which means that the meson-nucleon-coupling is weak rather than strong. Wentzel [same Acta 18, 430–446 (1945); these Rev. 7, 182] pointed out that the isobar states can have an appreciable influence on the neutron-proton scattering, even producing forward instead of backward scattering in a symmetrical theory. This effect, however, becomes immaterial because of the above-mentioned high excitation energy. *L. Hulthén* (Lund).

Valatin, J. Discussion of diatomic molecules without the two-center model. Mat. Fiz. Lapok 50, 115–161 (1943). (Hungarian. English summary)

Kovács, I. Über die Grundlagen der Theorie des zweiatomigen Moleküls. Mat. Fiz. Lapok 48, 334–350 (1941). (Hungarian. German summary)

Die vorliegende Arbeit stellt ein Referat über die Aufstellung und Eigenschaften der Wellengleichung des zweiatomigen Moleküls dar. Bei Vernachlässigung gewisser Glieder lässt sich die Wellengleichung in drei Teile separieren, in dem die Wellenfunktion als Produkt von drei Funktionen dargestellt wird. Die erste von diesen enthält als Variabeln die Ort- und Spinkoordinaten der einzelnen Elektronen, die zweite den Kernabstand, die dritte die Koordinaten der Molekülachse.

From the author's summary.

***Heisenberg, W., editor. Cosmic Radiation. Fifteen Lectures.** Translated by T. H. Johnson. Dover Publications, New York, N. Y., 1946. ix+192 pp. $3.50.

[The German edition was published by Springer, Berlin, 1943.] This book consists of articles which are the result of symposia on cosmic radiation held in 1941 and 1942 at the Kaiser Wilhelm Institute for Physics and give a general picture of the state of the field at that time. The lectures are grouped into five parts. In the introduction Heisenberg gives a short review of the whole field. The second part is devoted to cascades and consists of a lecture by Heisenberg on cascade theory and one by Molière on large air showers. The third part, on the meson, contains nine articles on the various properties of this particle and the effects it produces in the atmosphere. The fourth part, on nuclear particles, contains an article by Bagge on nuclear disruptions and one by Flügge on the neutrons in the atmosphere. The last part consists of an article by Meixner on the geomagnetic effects. For the most part the articles deal with reviews of existing experimental results. Only a few are on theoretical work and even in these lectures the results are usually quoted with only brief descriptions of the method of calculation.

Lectures which contain previously unpublished work are the one by Heisenberg on cascade theory, one by Molière on large showers, and one by Flügge on the neutron distribution in the atmosphere. In the lecture on cascade theory, Heisenberg gives a simplified theory of shower formation which follows the line of attack of Landau and Rumer. By making some simplifying assumptions about the radiation cross sections and in the calculations, he derives approximate formulae for the position of the cascade maximum and for the number of electrons and light quanta at this point. More accurate calculations for these quantities have been carried out by several authors but the method given here is much simpler. Molière reports on his work in which he improves the calculations of Euler and Wergeland on the spatial and angular distribution of shower particles. Following the work of Landau, Molière calculates the mean square of the spatial and angular deviations in a shower without neglecting such effects as that of large statistical fluctuations and the contributions of earlier generations to the deflections of shower particles of later generations. The new results give a greater spatial extension of the shower and a stronger branching of the core of the shower. Flügge's article on the neutrons in the atmosphere contains a calculation of the intensity of slow neutrons as a function of the altitude. A source of fast neutrons due to nuclear "stars" is assumed which decreases exponentially with increasing atmospheric pressure and which is also distributed exponentially in energy and the slowing down process is calculated by elementary diffusion theory.

Other lectures which are of theoretical nature are the three on meson theory and the last one on the geomagnetic theory. In the first, von Weizsäcker gives the fundamental ideas of Yukawa's attempt to give a unified picture of nuclear forces and cosmic-ray mesons, and discusses the difficulties which have been encountered. This work is continued by Flügge who considers, in particular, the attempts to explain the properties of the deuteron by various types of meson theory of nuclear forces. Then Heisenberg discusses the limitations of the present formulation of the quantum field theories and suggests the possibility of explosion-like showers at high energies. Meixner's article reviews the geomagnetic theory of cosmic rays developed by Störmer, Lemaître, Vallarta and their collaborators, and gives brief descriptions of the main results.

Although written by many authors, the articles are well integrated with numerous cross references and a self-consistent notation. The translation is excellent.

S. Kusaka (Princeton, N. J.).

BIBLIOGRAPHICAL NOTES

***Barnouw, A. J., and Landheer, B., editors. The Contribution of Holland to the Sciences.** Querido, New York, 1943. xviii+373 pp. (8 plates)

Contains chapters on Mathematics by D. J. Struik, Astronomy by J. Schilt and Physics by S. A. Goudsmit and I. B. Cohen.

Association Française pour l'Avancement des Sciences. 64^{me} Session, Paris, 20–26 Octobre 1945. **Mathématiques.** 142 pp.

Contains 40 lectures and abstracts. Issued as a supplement to L'Intermédiaire des Recherches Mathématiques, no. 9 (January, 1947).